Dr. Z's Math251 Handout #14.8 (2nd ed.) [Lagrange Multipliers: Optimizing with a Constraint]

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Problem Type 14.8a: Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given conditions.

$$f(x,y,z) = Expression_(x,y,z) \quad ; \quad g(x,y,z) = k \quad .$$

Example Problem 14.8a: Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given conditions.

$$f(x,y,z) = xyz \quad ; \quad 3x^2 + 2y^2 + z^2 = 6 \quad .$$

Steps

1. Find the gradients of $f(x,y,z)$ and $g(x,y,z)$.

Example

1. $f_x = \frac{\partial}{\partial x} xyz = yz$, $f_y = \frac{\partial}{\partial y} xyz = xz$, $f_z = \frac{\partial}{\partial z} xyz = xy$. So,

$$\bigtriangledown f = \langle yz, xz, xy \rangle \quad .$$

$g_x = \frac{\partial}{\partial x}(3x^2 + 2y^2 + z^2) = 6x$, $g_y = \frac{\partial}{\partial y}(3x^2 + 2y^2 + z^2) = 4y$, $g_z = \frac{\partial}{\partial z}(3x^2 + 2y^2 + z^2) = 2z$. So

$$\bigtriangledown g = \langle 6x, 4y, 2z \rangle \quad .$$

2. Introduce another variable, λ, and set up the equations implied by

$$\bigtriangledown f = \lambda \bigtriangledown g \quad .$$

To them add, the equation $g(x,y,z) = k$.

2. $\bigtriangledown f = \lambda \bigtriangledown g$ means

$$\langle yz, xz, xy \rangle = \lambda \langle 6x, 4y, 2z \rangle \quad ,$$

that spells-out to the set of equations

$$yz = 6\lambda x \quad , \quad xz = 4\lambda y \quad ,$$

$$xy = 2\lambda z \quad , \quad 3x^2 + 2y^2 + z^2 = 6 \quad .$$

3. Use algebra to solve the system of four equations and four unknowns.

3. Multiplying the first three equations, we get $(xyz)^2 = 48\lambda^3 xyz$ so $xyz = 48\lambda^3$, and hence $yz = 48\lambda^3/x$ and we get from the first equation $48\lambda^3/x = 6\lambda x$. This means $8\lambda^2 = x^2$ and so $x = \sqrt{8}\lambda$.

From $xyz = 48\lambda^3$, we also get $xz = 48\lambda^3/y$ and we get from the second equation $48\lambda^3/y = 4\lambda y$. This means $12\lambda^2 = y^2$ and so $y = \sqrt{12}\lambda$.

From $xyz = 48\lambda^3$, we also get $xy = 48\lambda^3/z$ and we get from the third equation $48\lambda^3/z = 2\lambda z$. This means $24\lambda^2 = z^2$ and so $z = \sqrt{24}\lambda$.

Plugging these expressions in λ into the last equation $3x^2 + 2y^2 + z^2 = 6$ we get

$$3(\sqrt{8}\lambda)^2 + 2(\sqrt{12}\lambda)^2 + (\sqrt{24}\lambda)^2 = 6 \quad .$$

So,

$$(3 \cdot 8 + 2 \cdot 12 + 24)\lambda^2 = 6 \quad ,$$

$$\lambda^2 = \frac{1}{12} \quad ,$$

and so

$$\lambda = \pm\frac{1}{\sqrt{12}} \quad .$$

We get **two** solutions. The first one is

$$\lambda = \frac{1}{\sqrt{12}} \quad , \quad x = \sqrt{8} \cdot \frac{1}{\sqrt{12}} = \sqrt{2/3} \quad ,$$

$$y = \sqrt{12} \cdot \frac{1}{\sqrt{12}} = 1 \quad , \quad z = \sqrt{24} \cdot \frac{1}{\sqrt{12}} = \sqrt{2} \quad ,$$

which means the point $(\sqrt{2/3}, 1, \sqrt{2})$.

And the second is

$$\lambda = \frac{-1}{\sqrt{12}} \quad , \quad x = \sqrt{8} \cdot \frac{-1}{\sqrt{12}} = -\sqrt{2/3} \quad ,$$

$$y = \sqrt{12} \cdot \frac{-1}{\sqrt{12}} = -1 \quad , z = \sqrt{24} \cdot \frac{-1}{\sqrt{12}} = -\sqrt{2} \quad ,$$

which means the point $(-\sqrt{2/3}, -1, -\sqrt{2})$.

4. Now you can forget about the λ and plug-in these point(s) into f and see who gives the largest value, that is the **maximum value** and who is the smallest, that is the **minimum value**.

4.

$$f(\sqrt{2/3}, 1, \sqrt{2}) = 2/\sqrt{3} \quad ,$$

$$f(-\sqrt{2/3}, -1, -\sqrt{2}) = -2/\sqrt{3} \quad .$$

Ans.: The maximum value is $2/\sqrt{3}$ and the minimum value is $-2/\sqrt{3}$.

A Problem from a Previous Final

Use Largange multipliers (no credit for other methods) to find the largest value that $x + 3y + 5z$ can be, given that $x^2 + y^2 + z^2 = 35$

Ans.: 35.