**Dr. Z's Math251 Handout #14.6 (2nd ed.) [The Chain Rule]**

By Doron Zeilberger

**Problem Type 14.6a**: Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ :

$$z = f(x,y) \quad , \quad x = h_1(s,t) \quad , \quad y = h_2(s,t) \quad .$$

**Example Problem 14.6a**: Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ :

$$z = x^3 + 2xy + y^2 \quad , \quad x = s + 2t \quad , \quad y = s^2 t \quad .$$

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**Steps**

**1.** Set-up the chain-rule:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad ,$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \quad .$$

**2.** For the specific functions given compute all the necessary quantities that show up on the right sides, namely: $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial x}{\partial s}$, $\frac{\partial y}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial t}$.

**3.** Incorporate them into the fomulas of step 1.

**Example**

**1.** Ditto.

**2.**

$$\frac{\partial z}{\partial x} = 3x^2 + 2y \quad , \quad \frac{\partial z}{\partial y} = 2x + 2y \quad .$$

$$\frac{\partial x}{\partial s} = 1 \quad , \quad \frac{\partial y}{\partial s} = 2st \quad .$$

$$\frac{\partial x}{\partial t} = 2 \quad , \quad \frac{\partial y}{\partial t} = s^2 \quad .$$

**3.**

$$\frac{\partial z}{\partial s} = (3x^2+2y)\cdot 1 + (2x+2y)(2st) = 3x^2+2y+4xst+4yst \quad .$$

$$\frac{\partial z}{\partial t} = (3x^2+2y)\cdot 2 + (2x+2y)s^2 = 6x^2+4y+2xs^2+2ys^2 \quad .$$

**Ans.:** $\frac{\partial z}{\partial s} = 3x^2 + 2y + 4xst + 4yst$, $\frac{\partial z}{\partial t} = 6x^2 + 4y + 2xs^2 + 2ys^2$.

**Note:** It is acceptable to have $x, y$ featuring in the answers as well as $t, s$. If you are asked to express your answers **only** in terms of the "basic" variables $t, s$, then you would have to do an extra step: substituting everywhere you see $x$ or $y$ their expressions in terms of $t$ and $s$. So in that case, the answers would be

$$\frac{\partial z}{\partial s} = 3(s+2t)^2 + 2s^2t + 4(s+2t)st + 4(s^2t)st \quad ,$$

$$\frac{\partial z}{\partial t} = 6(s+2t)^2 + 4s^2t + 2(s+2t)s^2 + 2(s^2t)s^2 \quad .$$

**Do not bother to expand** (unless specifically asked to).

**A Problem from a previous Final**

Find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial s}$ as **functions of** $r$ **and** $s$ , if

$$f(x,y) = x^3 + 2xy + y^3 \quad ,$$

and the variables are related by $x = r - s$ and $y = r + s$. You do not need to simplify!

**Ans.** $\frac{\partial f}{\partial r} = 3(r-s)^2+2(r+s)+3(r+s)^2+2(r-s)$, $\frac{\partial f}{\partial s} = -3(r-s)^2-2(r+s)+3(r+s)^2+2(r-s)$.

**Another Problem from a Previous Final**

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$\sin(x+2y+3z) = 5xyz + 1 \quad .$$

**Ans.** $(5yz - \cos(x+2y+3z))/(3\cos(x+2y+3z) - 5xy)$; $(5xz - 2\cos(x+2y+3z))/(3\cos(x+2y+3z) - 5xy)$ .