**Dr. Z's Math251 Handout #14.5 (2nd ed.) [The Gradient and Directional Derivatives]**

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**Problem Type 14.5a**: Find the directional derivative of the function $f(x, y, z)$ at the point $(x_0, y_0, z_0)$ in the direction $\langle v_1, v_2, v_3 \rangle$.

**Example Problem 14.5a**: Find the directional derivative of the function $f(x, y, z) = \ln(x^2 + y^2 + z^2)$ at the point $(2, 1, 3)$ in the direction $\langle 1, 2, 2 \rangle$.

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**Steps**

**1.** Find the **gradient** $\nabla f = \langle f_x, f_y, f_z \rangle$ by taking all the first partial derivatives. Also find the unit vector in the direction of $\langle v_1, v_2, v_3 \rangle$ by dividing by its length.

**2.** Plug-in $x = x_0, y = y_0, z = z_0$ into $\nabla f$ .

**Example**

**1.**

$$f_x = \frac{2x}{x^2 + y^2 + z^2} \ , \ f_y = \frac{2y}{x^2 + y^2 + z^2} \ , \ f_z = \frac{2z}{x^2 + y^2 + z^2} \quad .$$

So

$$\nabla f = \langle \frac{2x}{x^2 + y^2 + z^2}, \frac{2y}{x^2 + y^2 + z^2}, \frac{2z}{x^2 + y^2 + z^2} \rangle$$

$|\langle 1, 2, 2 \rangle| = \sqrt{1^2 + 2^2 + 2^2} = 3$, so

$$\mathbf{u} = \frac{1}{3} \langle 1, 2, 2 \rangle = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle \quad .$$

**2.**

$$\nabla f(2, 1, 3) = \langle \frac{2 \cdot 2}{2^2 + 1^2 + 3^2}, \frac{2 \cdot 1}{2^2 + 1^2 + 3^2}, \frac{2 \cdot 3}{2^2 + 1^2 + 3^2} \rangle$$

$$= \langle \frac{2}{7}, \frac{1}{7}, \frac{3}{7} \rangle \quad .$$

**3.** Take the dot product $\bigtriangledown f.\mathbf{u}$ .

**3.**

$$\langle \frac{2}{7}, \frac{1}{7}, \frac{3}{7} \rangle . \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$$

$$= \frac{2}{7} \cdot \frac{1}{3} + \frac{1}{7} \cdot \frac{2}{3} + \frac{3}{7} \cdot \frac{2}{3} = \frac{10}{21}$$

**Ans.:** The requested directional derivative is $\frac{10}{21}$.

**Problem Type 14.5b**: Find the maximum rate of change of $f$ at the given point and the direction in which it occurs.

$$f(x, y) = Expression(x, y) \quad , \quad (x_0, y_0) \quad .$$

**Example Problem 14.5b**: Find the maximum rate of change of $f$ at the given point and the direction in which it occurs.

$$f(x, y) = \sin(xy) \quad , \quad (1, 0).$$

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**Steps**

**Example**

**1.** Find the gradient

$$\bigtriangledown f = \langle f_x, f_y \rangle$$

**1.** $f_x = y\cos(xy), f_y = x\cos(xy)$. So

$$\bigtriangledown f = \langle f_x, f_y \rangle = \langle y\cos(xy), x\cos(xy) \rangle \quad .$$

**2.** Plug-in $x = x_0, y = y_0$ into $\bigtriangledown f$.

**2.**

$$\bigtriangledown f(1, 0) = \langle 0 \cdot \cos(0), 1 \cdot \cos(0) \rangle = \langle 0, 1 \rangle \quad .$$

**3.** The maximum rate of change of $f$ is simply the length of $\bigtriangledown f$ at the designated point. The direction in which is occurs is that direction. So find the unit vector in that direction.

**3.**

$$|\langle 0, 1 \rangle| = \sqrt{0^2 + 1^2} = 1 \quad .$$

$\langle 0, 1 \rangle$ is already a unit vector, so the direction is $\langle 0, 1 \rangle$.

**Ans.**: The maximum rate of change is 1 in the direction $\langle 0, 1 \rangle$ (or $\mathbf{j}$ ).

**A Problem from a Previous Final**

Let

$$f(x,y,z) = -x^2 + y^2 + z^2 - 1 \quad .$$

**(a)** (2 points) Compute $\bigtriangledown f$.

**(b)** (5 points) Find a normal to the level surface $f(x,y,z) = 0$ at the point $(1,1,1)$, and give an equation for the tangent plane to that surface at that point.

**(c)** (6 points) Compute the directional derivative of $f(x,y,z)$ at the point $(1,1,1)$ in the direction $\langle 1 \, , \, 2 \, , \, 2 \rangle$ .

**Ans.**: a) $\langle -2x, 2y, 2z \rangle$ ; b) $z = x - y + 1$; c) 2.