Dr. Z's Math251 Handout #13.5 (2nd ed.) [Motion in Three-Space]

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Problem Type 13.5a: Find the velocity, acceleration, and speed of a particle with the given position function.

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \quad .$$

Example Problem 13.5a: Find the velocity, acceleration, and speed of a particle with the given position function.

$$\mathbf{r}(t) = t^2\,\mathbf{i} + \ln t\,\mathbf{j} + t\,\mathbf{k} \quad .$$

Steps

1. The velocity is $\mathbf{r}'(t)$ and the acceleration is $\mathbf{r}''(t)$.

Example

1. $\mathbf{v}(t) = \mathbf{r}'(t) = (t^2)'\,\mathbf{i} + (\ln t)'\,\mathbf{j} + t'\,\mathbf{k} = 2t\,\mathbf{i} + 1/t\,\mathbf{j} + \mathbf{k}$.

$\mathbf{a}(t) = \mathbf{r}''(t) = (2t)'\,\mathbf{i} + (1/t)'\,\mathbf{j} + 1'\,\mathbf{k} = 2\,\mathbf{i} - (1/t^2)\,\mathbf{j} \quad .$

Steps

2. To find the speed, take the magnitude of the velocity, i.e. compute $|\mathbf{v}(t)|$

Example

2. the speed is $|\mathbf{v}(t)| = |2t\,\mathbf{i} + 1/t\,\mathbf{j} + \mathbf{k}| = \sqrt{(2t)^2 + (1/t)^2 + 1^2} = \sqrt{4t^2 + 1/t^2 + 1}$.

Problem Type 13.5b: A force with magnitute FN acts on a body of mass m in the directon $\langle d_1, d_2, d_3 \rangle$. The object starts at the (x_0, y_0, z_0) with initial velocity $\mathbf{v}(0) = \langle v_1, v_2, v_3 \rangle$. Find its position function and its speed at time t.

Example Problem 13.5b: A force with magnitute $300N$ acts on a body of mass 100 kg in the directon $\langle 1, 2, 2 \rangle$. The object starts at the $(1, 2, 3)$ with initial velocity $\mathbf{v}(0) = \langle 0, 1, -1 \rangle$. Find its position function and its speed at time t.

Steps

1. Find the **unit direction vector** by dividing $\langle d_1, d_2, d_3 \rangle$ by its length. To get the **force** vector, multiply this vector by the magnitude of the force F.

Example

1. $|\langle 1, 2, 2 \rangle| = \sqrt{1^2 + 2^2 + 2^2} = 3$. So the direction of the force is $(1/3)\langle 1, 2, 2 \rangle = \langle 1/3, 2/3, 2/3 \rangle$. The **force is** $\mathbf{F} = 300\langle 1/3, 2/3, 2/3 \rangle$, so $\mathbf{F} = \langle 100, 200, 200 \rangle$.

2. Set-up **Newton's Second Law**

$$\mathbf{F} = m\mathbf{r}''(t) \quad .$$

2.

$$\langle 100, 200, 200 \rangle = 100\mathbf{r}''(t) \quad .$$

so

$$\mathbf{r}''(t) = \langle 1, 2, 2 \rangle \quad .$$

3. Integrate with respect to t to get $\mathbf{v}(t) = \mathbf{r}'(t)$, and don't forget the **arbitrary constant** vector. Then plug-in $t = 0$ to get it.

3.

$$\mathbf{v}(t) = \mathbf{r}'(t) = \int \langle 1, 2, 2 \rangle \, dt = \langle t, 2t, 2t \rangle + \mathbf{C} \quad .$$

But $\mathbf{v}(0) = \langle 0, 1, -1 \rangle$, so $\mathbf{C} = \langle 0, 1, -1 \rangle$ and we get

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle t, 2t, 2t \rangle + \langle 0, 1, -1 \rangle = \langle t, 2t+1, 2t-1 \rangle \quad .$$

4. To get the position vector $\mathbf{r}(t)$ integrate the velocity vector $\mathbf{v}(t)$ that you found in step 2, once again not forgetting the arbitrary constant vector.

4.

$$\mathbf{r}(t) = \int \langle t, 2t+1, 2t-1 \rangle \, dt = \langle t^2/2, t^2+t, t^2-t \rangle + \mathbf{C}$$

When $t = 0$, $\mathbf{r}(0) = \langle 1, 2, 3 \rangle$, so $\mathbf{C} = \langle 1, 2, 3 \rangle$, and

$$\mathbf{r}(t) = \langle t^2/2, t^2+t, t^2-t \rangle + \langle 1, 2, 3 \rangle = \langle t^2/2+1, t^2+t+2, t^2-t+3 \rangle \quad .$$

This is the **position function**.

5. To get the **speed**, compute $|\mathbf{v}(t)|$, an expression in t.

5.

$$|\mathbf{v}(t)| = |\langle t, 2t+1, 2t-1 \rangle|$$

$$= \sqrt{t^2 + (2t+1)^2 + (2t-1)^2} = \sqrt{9t^2 + 2} \quad .$$