Dr. Z's Math251 Handout #13.2 (2nd ed.) [Calculus of Vector-Valued Functions]

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Problem Type 13.2a: Find a parametric equation for the tangent line to the curve with the given parametric equation at the specified point

$$x = f_1(t)\,,\, y = f_2(t)\,,\, z = f_3(t) \quad ; \quad P(p_1, p_2, p_3)$$

Example Problem 13.2a: Find a parametric equation for the tangent line to the curve with the given parametric equation at the specified point

$$x = t^2 - 1 \quad , \quad y = t^2 + 1 \quad , \quad z = t + 1 \quad ; \quad (-1, 1, 1)$$

Steps	**Example**
1. Find the relevant t at the designated point. Let's call it t_0. Solve the equations $f_1(t_0) = p_1, f_2(t_0) = p_2, f_3(t_0) = p_3$. If there is no solution refuse to do the problem.	**1.** $-1 = t_0^2 - 1\,,\, 1 = t_0^2 + 1\,,\, 1 = t_0 + 1$ means $t_0 = 0$.
2. Putting $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, take the derivative $\mathbf{r}'(t)$ by doing it component-by-component $$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle \quad .$$	**2.** $$\mathbf{r}(t) = \langle\, t^2 - 1\,,\, t^2 + 1\,,\, t + 1\,\rangle$$ $$\mathbf{r}'(t) = \langle\, (t^2-1)'\,,\, (t^2+1)'\,,\, (t+1)'\,\rangle = \langle\, 2t\,,\, 2t\,,\, 1\,\rangle \quad .$$
3. Plug-in the specific value ($t = t_0$), that you found in step 1, to get $\mathbf{r}'(t_0)$, which is the **direction vector**, let's call it $\mathbf{D}$. and using the given point P as the **starting point**, the **parametric equation** of the line is $\langle x, y, z\rangle = P + tD$. Finally, spell out the expressions for x, y, z.	**3.** The direction vector is $\langle\, 2t\,,\, 2t\,,\, 1\,\rangle$ plugged-in at $t = 0$, it is $\langle 0\,,\, 0\,,\, 1\,\rangle$, and since the point is $\langle\, -1\,,\, 1\,,\, 1\,\rangle$ the equation of the tangent line, in vector-form is $$\langle\, -1\,,\, 1\,,\, 1\,\rangle + t\langle\, 0\,,\, 0\,,\, 1\,\rangle = \langle\, -1\,,\, 1\,,\, 1+t\,\rangle \quad ,$$ and spelling it out $$x = -1 \quad , \quad y = 1 \quad , \quad z = 1 + t \quad .$$ This is the **Ans.**.

Problem Type 13.2b: Find $\mathbf{r}(t)$ if

$$\mathbf{r}'(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$$

and

$$\mathbf{r}(t_0) = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$$

Example Problem 13.2b: Find $\mathbf{r}(t)$ if

$$\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + 4t^3\,\mathbf{k}$$

and

$$\mathbf{r}(1) = \mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}$$

Steps

1. Find the indefinite integral by integrating every component and not forgetting to add **an arbitrary constant** that now is a **vector**.

2. Plug-in $t = t_0$ and solve for $\mathbf{C}$.

3. Go back to step 1 and incorporate the specific $\mathbf{C}$ that you've found in step 2.

Example

1.

$$\mathbf{r}(t) = \int (2t\,\mathbf{i} + 3t^2\,\mathbf{j} + 4t^3\,\mathbf{k})\,dt$$

$$= t^2\,\mathbf{i} + t^3\,\mathbf{j} + t^4\,\mathbf{k} + \mathbf{C} \quad .$$

2.

$$\mathbf{r}(1) = 1^2\,\mathbf{i} + 1^3\,\mathbf{j} + 1^4\,\mathbf{k} + \mathbf{C} =$$

$$\mathbf{i} + \mathbf{j} + \mathbf{k} + \mathbf{C} = \mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k} \quad .$$

Solving for $\mathbf{C}$ gives $\mathbf{C} = 3\,\mathbf{j} + 4\,\mathbf{k}$

3.

$$\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} + t^4\,\mathbf{k} + 3\,\mathbf{j} + 4\,\mathbf{k}$$

$$= t^2\,\mathbf{i} + (t^3 + 3)\,\mathbf{j} + (t^4 + 4)\,\mathbf{k} \quad .$$

This is the **Ans.**.

Problem from a previous Final

Find the velocity and position vectors of a particle whose acceleration is $\mathbf{a}(t) = \mathbf{i} + \mathbf{j}$, and time $t = 0$, the velocity is $\mathbf{i} - \mathbf{j}$ and position is $\mathbf{k}$.

Ans.

$$\mathbf{v}(t) = (t+1)\mathbf{i} + (t-1)\mathbf{j} \quad ; \quad \mathbf{r}(t) = (t^2/2 + t)\mathbf{i} + (t^2/2 - t)\mathbf{j} + \mathbf{k} \quad .$$