Dr. Z's Math251 Handout #12.5 (2nd ed.) [Planes in Three Space]

By Doron Zeilberger

**Problem Type 12.5a:** Find an equation of the plane that passes through three given points

**Example Problem 12.5a:** Find an equation of the plane that passes through the points \((1, 1, 1), (2, 0, 1), (2, 1, 0)\).

---

**Steps**

1. Calling these points \(P, Q, R\) find the vectors \(PQ\) and \(PR\) by doing \(Q - P\) and \(R - P\).

   \[
   \begin{align*}
P &= (1, 1, 1), \\
Q &= (2, 0, 1), \\
R &= (2, 1, 0),
\end{align*}
\]

   \[
   \begin{align*}
PQ &= Q - P = \langle 2 - 1, 0 - 1, 1 - 1 \rangle = \langle 1, -1, 0 \rangle, \\
PR &= R - P = \langle 2 - 1, 1 - 0, 0 - 1 \rangle = \langle 1, 0, -1 \rangle.
\end{align*}
\]

2. Find a vector normal to the plane by computing the cross-product \(PQ \times PR\).

   \[
   \begin{align*}
PQ \times PR &= \langle 1, -1, 0 \rangle \times \langle 1, 0, -1 \rangle = \\
   &= \begin{vmatrix}
i & j & k \\
1 & -1 & 0 \\
1 & 0 & -1
\end{vmatrix} = \\
   &= \begin{vmatrix}
i & -1 & 0 \\
0 & -1 & 1 \\
0 & 1 & 1
\end{vmatrix} + \begin{vmatrix}
i & 1 & 0 \\
-1 & 1 & -1 \\
1 & 0 & 1
\end{vmatrix} = \\
   &= i + j + k = \langle 1, 1, 1 \rangle.
\end{align*}
\]

This is the **normal vector** \(n = \langle a, b, c \rangle\).

So \(n = \langle a, b, c \rangle = \langle 1, 1, 1 \rangle\)
3. Pick any of the three points (it does not matter which) as the reference point \((x_0, y_0, z_0)\) and use the formula

\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 .
\]

3. Picking \(P\) we get \((x_0, y_0, z_0) = (1, 1, 1)\), since \(a = 1, b = 1, c = 1\), we get that an equation is

\[
1 \cdot (x - 1) + 1 \cdot (y - 1) + 1 \cdot (z - 1) = 0
\]

and expanding we get

\[
x + y + z = 3 .
\]

**Ans.:** An equation for the plane passing through \(P, Q\) and \(R\) is

\[
x + y + z = 3.
\]

**Check:** Plug-in all the three points into the equation and make sure that they agree.

**Problem Type 12.5b:** Find direction numbers for the line of intersections of the planes \(a_1x + b_1y + c_1z = d_1\) and \(a_2x + b_2y + c_2z = d_2\).

**Example Problem 12.5b:** Find direction numbers for the line of intersections of the planes

\[
2x + 3y + 4z = 2 \text{ and } -3x + 2y + 3z = 1.
\]

**Steps**

1. By looking at the coeffs. of \(x, y, z\) extract the **normal vectors** \(\mathbf{n}_1 = \langle a_1, b_1, c_1 \rangle\) and \(\mathbf{n}_2 = \langle a_2, b_2, c_2 \rangle\).

**Note:** the numbers on the right sides \((d_1, d_2)\) are not needed.

2. Take the cross-product \(\mathbf{n}_1 \times \mathbf{n}_2\). The components are the **direction numbers** of the line of intersection.

**Example**

1. \(\mathbf{n}_1 = \langle 2, 3, 4 \rangle\) and \(\mathbf{n}_2 = \langle -3, 2, 3 \rangle\).

2. \(\mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, 3, 4 \rangle \times \langle -3, 2, 3 \rangle = \langle 1, -18, 13 \rangle\).

(You do it!)

**Ans.:** The direction numbers are \(\langle 1, -18, 13 \rangle\).
Problem from a previous Final

Find an equation for the plane through the point $(1, 0, 2)$ that contains the line

$$
\mathbf{r}(t) = \langle 1, 1, 1 \rangle + t\langle 1, -1, 0 \rangle.
$$

Simplify as much as you can!

Ans.: $x + y + z = 3$. 