Dr. Z's Math251 Handout #12.5 (2nd ed.) [Planes in Three Space]

By Doron Zeilberger

Problem Type 12.5a: Find an equation of the plane that passes through three given points

Example Problem 12.5a: Find an equation of the plane that passes through the points $(1, 1, 1)$, $(2, 0, 1)$, $(2, 1, 0)$.

Steps

1. Calling these points P, Q, R find the vectors **PQ** and **PR** by doing $Q - P$ and $R - P$.

2. Find a vector normal to the plane by computing the cross-product $\mathbf{PQ} \times \mathbf{PR}$.

Example

1. $P = (1, 1, 1), Q = (2, 0, 1), R = (2, 1, 0)$,

$\mathbf{PQ} = Q - P = \langle 2-1, 0-1, 1-1 \rangle = \langle 1, -1, 0 \rangle$,

$\mathbf{PR} = R - P = \langle 2-1, 1-1, 0-1 \rangle = \langle 1, 0, -1 \rangle$.

2.

$$\mathbf{PQ} \times \mathbf{PR} = \langle 1, -1, 0 \rangle \times \langle 1, 0, -1 \rangle =$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} =$$

$$\mathbf{i} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= \mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1, 1, 1 \rangle \quad .$$

This is the **normal vector** $\mathbf{n} = \langle a, b, c \rangle$.
So $\mathbf{n} = \langle a, b, c \rangle = \langle 1, 1, 1 \rangle$

3. Pick any of the three points (it does not matter which) as the refence point (x_0, y_0, z_0) and use the formula

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad .$$

3. Picking P we get $(x_0, y_0, z_0) = (1, 1, 1)$, since $a = 1, b = 1, c = 1$, we get that an equation is

$$1 \cdot (x - 1) + 1 \cdot (y - 1) + 1 \cdot (z - 1) = 0$$

and expanding we get

$$x + y + z = 3 \quad .$$

Ans.: An equation for the plane passing through P, Q and R is $x + y + z = 3$.

Check: Plug-in all the three points into the equation and make sure that they agree.

Problem Type 12.5b: Find direction numbers for the line of intersections of the planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$.

Example Problem 12.5b: Find direction numbers for the line of intersections of the planes $2x + 3y + 4z = 2$ and $-3x + 2y + 3z = 1$.

Steps	**Example**
1. By looking at the coeffs. of x, y, z extract the **normal vectors** $\mathbf{n_1} = \langle a_1, b_1, c_1 \rangle$ and $\mathbf{n_2} = \langle a_2, b_2, c_2 \rangle$. **Note:** the numbers on the right sides (d_1, d_2) are not needed.	**1.** $\mathbf{n_1} = \langle 2, 3, 4 \rangle$ and $\mathbf{n_2} = \langle -3, 2, 3 \rangle$.
2. Take the cross-product $\mathbf{n_1} \times \mathbf{n_2}$. The components are the **direction numbers** of the line of intersection.	**2.** $\mathbf{n_1} \times \mathbf{n_2} = \langle 2, 3, 4 \rangle \times \langle -3, 2, 3 \rangle = \langle 1, -18, 13 \rangle \quad .$ (You do it!) **Ans.**: The direction numbers are $\langle 1, -18, 13 \rangle$.

Problem from a previous Final

Find an equation for the plane through the point $(1, 0, 2)$ that contains the line

$$\mathbf{r}(t) = \langle 1, 1, 1\rangle + t\langle 1, -1, 0\rangle \quad .$$

Simplify as much as you can!

Ans.: $x + y + z = 3$.