Dr. Z's Math251 Handout #12.4 (2nd ed.) [The Cross Product]

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Problem Type 12.4a: Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \quad , \quad \mathbf{b} = \langle b_1, b_2, b_3 \rangle \quad .$$

Example Problem 12.4a: Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a} = \langle 4, 5, -3 \rangle \quad , \quad \mathbf{b} = \langle 2, 1, -3 \rangle \quad .$$

Steps

1. Form the **determinant** with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the first row, the vector $\mathbf{a}$ in the second row and the vector $\mathbf{b}$ in the third row.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad .$$

2. Evaluate the determinant step-by-step:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$\mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \mathbf{i}(a_2b_3 - b_2a_3) - \mathbf{j}(a_1b_3 - b_1a_3) + \mathbf{k}(a_1b_2 - b_1a_2) \quad .$$

Example

1.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -3 \\ 2 & 1 & -3 \end{vmatrix} \quad .$$

2.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -3 \\ 2 & 1 & -3 \end{vmatrix} =$$

$$\mathbf{i} \begin{vmatrix} 5 & -3 \\ 1 & -3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 4 & -3 \\ 2 & -3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix}$$

$$= \mathbf{i}(5\cdot(-3) - (1)\cdot(-3)) - \mathbf{j}(4\cdot(-3) - (2)\cdot(-3)) + \mathbf{k}(4\cdot(1) - (2)\cdot(5)) =$$

$$-12\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$$

First Part of the Answer:

$-12\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$ or $\langle -12, 6, -6 \rangle$.

3. Take the dot product of this vector with **a** and **b** and verify that they are both 0. If you get something else, something went wrong.

3. $\langle -12, 6, -6\rangle.\langle 4, 5, -3\rangle = (-12)(4) + (6)(5) + (-6)(-3) = -48 + 30 + 18 = 0$

and

$\langle -12, 6, -6\rangle.\langle 2, 1, -3\rangle = (-12)(2)+(6)(1)+(-6)(-3) = -24 + 6 + 18 = 0.$

Second Part of Ans.: Both are zero as expected.

Problem Type 12.4b: (a) Find a vector orthogonal to the plane through the points $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$,$R = (r_1, r_2, r_3)$. (b) Find the area of the triangle PQR.

Example Problem 12.4b: (a) Find a vector orthogonal to the plane through the points $P = (2, 1, 5)$, $Q = (-1, 3, 4)$,$R = (3, 0, 6)$. (b) Find the area of the triangle PQR.

Steps

Example

1. Compute the vector **PQ** by subtracting Q from P and the vector **PR** by subtracting R from P.

1.

$\mathbf{PQ} = (-1, 3, 4)-(2, 1, 5) = \langle -3, 2, -1\rangle$,

$\mathbf{PR} = (3, 0, 6) - (2, 1, 5) = \langle 1, -1, 1\rangle$.

2. Take the cross product of **PQ** and **PR**.

$$\mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} .$$

$$= \mathbf{i}\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -3 & -1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= \mathbf{i} + 2\mathbf{j} + \mathbf{k} = \langle 1, 2, 1\rangle .$$

Ans. to First Part: A vector orthogonal to the plane of the triangle PQR is $\langle 1, 2, 1\rangle$.

3. Take the magnitude of the vector you found in step 2. This is the **area of the parralelogram**. To get the **area of the triangle**, divide it by 2.

3.

$$|\langle 1, 2, 1\rangle| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6} \quad .$$

So the area of the parallelogram formed by PQ and PR is $\sqrt{6}$ and the area of triangle PQR is half of that, $\sqrt{6}/2$.

Ans. to Second Part: The area of triangle PQR is $\sqrt{6}/2$.

Problem Type 12.4c: Find the volume of the parallelopiped with adjacent edges PQ,PR, and PS, where P, Q, R, S are given points.

Example Problem 12.4c: Find the volume of the parallelopiped with adjacent edges PQ,PR, and PS, where $P(0, 1, 2), Q(2, 4, 5), R(-1, 0, 1), S(6, -1, 4)$.

Steps

1. Compute the vectors **PQ**, **PR**,**PS** by the appropriate subtractions.

Example

1.

$$\mathbf{PQ} = Q - P = (2, 4, 5) - (0, 1, 2) = \langle 2, 3, 3\rangle \quad ,$$

$$\mathbf{PR} = R - P = (-1, 0, 1) - (0, 1, 2) = \langle -1, -1, -1\rangle \quad ,$$

$$\mathbf{PS} = S - P = (6, -1, 4) - (0, 1, 2) = \langle 6, -2, 2\rangle \quad .$$

2. Compute the cross product $\mathbf{PQ} \times \mathbf{PR}$.

2.

$$\mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 3 \\ -1 & -1 & -1 \end{vmatrix} \quad .$$

$$= \mathbf{i} \begin{vmatrix} 3 & 3 \\ -1 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix}$$

$$= 0 \cdot \mathbf{i} - (1) \cdot \mathbf{j} + (1) \cdot \mathbf{k} = -\mathbf{j} + \mathbf{k} = \langle 0, -1, 1 \rangle$$

3. Compute the dot product of what you found in step 2 with the vector $\mathbf{PS}$.

3.

$$(\mathbf{PQ} \times \mathbf{PR}) \cdot \mathbf{PS} = \langle 0, -1, 1 \rangle . \langle 6, -2, 2 \rangle =$$

$$(0) \cdot (6) + (-1) \cdot (-2) + (1) \cdot (2) = 4 \quad .$$

Ans.: The volume of the parallelopiped is 4.