Dr. Z's Math251 Handout (2nd ed.) #12.3 [Dot Product and the Angle Between Two Vectors]

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Problem Type 12.3a : Use vectors to decide whether the triangle with $P(p_1, p_2, p_3)$, $Q(q_1, q_2, q_3)$, $R(r_1, r_2, r_3)$, is right-angled.

Example Problem 12.3a: Use vectors to decide whether the triangle with $P(2, -6, -4)$, $Q(4, 0, -8)$, $R(12, -4, -10)$, is right-angled.

Steps

1. Form the vectors PQ, QR, and RP by subtraction.

$$\mathbf{PQ} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle,$$

$$\mathbf{QR} = \langle r_1 - q_1, r_2 - q_2, r_3 - q_3 \rangle,$$

$$\mathbf{RP} = \langle p_1 - r_1, p_2 - r_2, p_3 - r_3 \rangle \quad .$$

2. Take all three dot products, and see whether any of them is 0. If this is the case, then it is indeed right-angled, otherwise not.

Example

1.

$$\mathbf{PQ} = \langle 4-2, 0-(-6), -8-(-4) \rangle = \langle 2, 6, -4 \rangle,$$

$$\mathbf{QR} = (12-4, -4-0, -10-(-8)) = \langle 8, -4, -2 \rangle,$$

$$\mathbf{RP} = (2-12, (-6)-(-4), (-4)-(-10)) = \langle -10, -2, 6 \rangle \quad .$$

2.

$$\mathbf{PQ}.\mathbf{QR} = \langle 2, 6, -4 \rangle . \langle 8, -4, -2 \rangle =$$

$$2\cdot 8 + 6\cdot(-4) + (-4)\cdot(-2) = 16-24+8 = 0 \quad ,$$

$$\mathbf{QR}.\mathbf{RP} = \langle 8, -4, -2 \rangle . \langle -10, -2, 6 \rangle =$$

$$= 8\cdot(-10) + (-4)\cdot(-2) + (-2)\cdot 6 = -80+8-12 = -84 \quad ,$$

$$\mathbf{PQ}.\mathbf{RP} = \langle 2, 6, -4 \rangle . \langle -10, -2, 6 \rangle =$$

$$= 2\cdot(-10) + 6\cdot(-2) + (-4)\cdot 6 = -20-12-24 = -56 \quad .$$

Since one of these dot-products is 0 it is indeed a **right-angled** triangle. Since the common vertex where it happens is Q the right-angle is at vertex Q.

Ans.: It is a right-angled triangle since one of the dot-products is 0.

Problem Type 12.3b: A constant force with vector representation $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ moves an object along a straight line from the point (p_1, p_2, p_3) to the point (q_1, q_2, q_3). Find the work done if the distance is measured in meters and the magnitude of the force is measured in newtons.

Example Problem 12.3b: A constant force with vector representation $\mathbf{F} = 5\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$ moves an object along a straight line from the point $(4, 6, 0)$ to the point $(8, 18, 30)$. Find the work done if the distance is measured in meters and the magnitude of the force is measured in newtons.

Steps	**Example**
1. Find the **displacement vector D** that connects the starting point to the endpoint: $$\mathbf{D} = (q_1 - p_1)\mathbf{i} + (q_2 - p_2)\mathbf{j} + (q_3 - p_3)\mathbf{k}$$	**1.** $$\mathbf{D} = (8-4)\mathbf{i} + (18-6)\mathbf{j} + (30-0)\mathbf{k}$$ $$= 4\mathbf{i} + 12\mathbf{j} + 30\mathbf{k} \quad .$$
2. Take the the dot product $\mathbf{D}.\mathbf{F}$.	**2.** $$\mathbf{D}.\mathbf{F} = (4\mathbf{i} + 12\mathbf{j} + 30\mathbf{k}).(5\mathbf{i} + 9\mathbf{j} - 3\mathbf{k})$$ $$= 4{\cdot}5+12{\cdot}9+30{\cdot}(-3) = 20+108-90 = 38 \quad .$$ **Ans.:** The work done was 38 joules.