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**Problem Type 12.2a:** Show that the triangle with vertices $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$, $R = (r_1, r_2, r_3)$ is an equilateral triangle.

**Example Problem 12.2a:** Show that the triangle with vertices $P = (-4, 8, 0)$, $Q = (2, 4, -2)$, $R = (-2, 2, 4)$ is an equilateral triangle.

**Steps**

1. Use the distance formula

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

for the distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, to find the three distances $|PQ|$, $|PR|$, $|QR|$.

**Example**

1. Here $P = (-4, 8, 0)$, $Q = (2, 4, -2)$, $R = (-2, 2, 4)$, so

$$|PQ| = \sqrt{(2 - (-4))^2 + (4 - 8)^2 + ((-2) - 0)^2} = \sqrt{36 + 16 + 4} = \sqrt{56}.$$

$$|PR| = \sqrt{((-2) - 4)^2 + (2 - 8)^2 + (4 - 0)^2} = \sqrt{4 + 36 + 16} = \sqrt{56}.$$

$$|QR| = \sqrt{((-2) - 2)^2 + (2 - 4)^2 + ((-4) - (-2))^2} = \sqrt{16 + 4 + 36} = \sqrt{56}.$$

2. Check whether these three distances are all the same. It there are, it is an equilateral triangle, otherwise not.

2. All the distances are the same ($\sqrt{56}$).

**Ans.:** It is an equilateral triangle since all the sides have equal length, namely: $\sqrt{56}$. 

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Dr. Z’s Math251 Handout (2nd ed.) #12.2 [Vectors in three dimensions]
Problem Type 12.2b: Find an equation of the sphere with center \( C(h, k, l) \) and radius \( r \).

Example Problem 12.2b: Find an equation of the sphere with center \((1, 2, -1)\) and radius 2.

**Steps**

1. Implement the formula

\[
(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2
\]

2. Expand everything and move everything to the left leaving 0 at the right side. Also rearrange terms so that the quadratic terms come before the linear terms.

**Example**

1. In this problem \((h, k, l) = (1, 2, -1)\) and \(r = 2\) so the equation is:

\[
(x - 1)^2 + (y - 2)^2 + (z - (-1))^2 = 2^2
\]

which is the same as

\[
(x - 1)^2 + (y - 2)^2 + (z + 1)^2 = 2^2
\]

2. Expand everything and move everything to the left leaving 0 at the right side.

\[
x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 + 2z + 1 = 4
\]

Cleaning up:

\[
x^2 + y^2 + z^2 - 2x - 4y + 2z + 2 = 0
\]

**Ans.**: \(x^2 + y^2 + z^2 - 2x - 4y + 2z + 2 = 0\).
Problem Type 12.2c: Show that the equation represents a sphere, and find the center and radius.

\[ x^2 + y^2 + z^2 + ax + by + cz + d = 0. \]

Example Problem 12.2c: Show that the equation represents a sphere, and find the center and radius.

\[ x^2 + y^2 + z^2 - 2x - 4y + 2z + 2 = 0. \]

Steps

1. The coefficients of \( x^2, y^2, z^2 \) should all be the same! If they are not, for example, if the equation is \( x^2 + y^2 + 3z^2 + 2x + 6y - 5 + 11 = 0 \) where the coefficients are not all the same, then it is **not** a sphere. Usually they are all 1, If the coefficient of \( x^2 \) (=coeff. of \( y^2 \)=coeff. of \( z^2 \)) is not 1, divide the whole equation by that coefficient, making it 1. The coeffs. of \( x^2, y^2, z^2 \) should now be all 1. Now group the terms so that \( x^2 \) is next to the \( x \) term, \( y^2 \) is next to the \( y \) term, and \( z^2 \) is next to the \( z \) term.

2. For each part separately, *complete the square*, using \( X^2 + aX = (X + a/2)^2 - (a/2)^2 \)

Example

1. In this problem, the coeffs. of \( x^2 \) is already 1, as are those of \( y^2 \) and \( z^2 \). Grouping the \( x \)-terms, \( y \)-terms and \( z \)-terms, we get:

\[ x^2 - 2x + y^2 - 4y + z^2 + 2z + 2 = 0 \]

2. 

\[
\begin{align*}
x^2 - 2x &= (x - 1)^2 - 1 \\
y^2 - 4y &= (y - 2)^2 - 4 \\
z^2 + 2z &= (z + 1)^2 - 1
\end{align*}
\]

So

\[
 x^2 - 2x + y^2 - 4y + z^2 + 2z + 2 = 0
\]

becomes

\[
(x-1)^2-1+(y-2)^2-4+(z+1)^2-1+2 = 0
\]
3. Move all the numbers to the right and express the resulting number on the right as $r^2$. Compare with the equation of the sphere

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2,$$

and read-off the center $(h, k, l)$ and the radius $r$.

3.

$$(x - 1)^2 - 1 + (y - 2)^2 - 4 + (z + 1)^2 - 1 + 2 = 0$$

is the same as

$$(x - 1)^2 + (y - 2)^2 + (z + 1)^2 = 4$$

which, in turn, is the same as

$$(x - 1)^2 + (y - 2)^2 + (z - (-1))^2 = 2^2,$$

which is an equation of a sphere with center $(1, 2, -1)$ and radius 2.