Dr. Z's Math251 Handout (2nd ed.) #12.2 [Vectors in three dimensions]

By Doron Zeilberger

Problem Type 12.2a: Show that the triangle with vertices $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$, $R = (r_1, r_2, r_3)$ is an equilateral triangle.

Example Problem 12.2a: Show that the triangle with vertices $P = (-4, 8, 0)$, $Q = (2, 4, -2)$, $R = (-2, 2, 4)$ is an equilateral triangle.

Steps

1. Use the distance formula

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad ,$$

for the distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, to find the three distances $|PQ|$, $|PR|$, $|QR|$.

2. Check whether theese three distances are all the same. It there are, it is an equilateral triangle, otherwise not.

Example

1. Here $P = (-4, 8, 0)$, $Q = (2, 4, -2)$, $R = (-2, 2, 4)$, so

$$|PQ| = \sqrt{(2 - (-4))^2 + (4 - 8)^2 + ((-2) - 0)^2} = \sqrt{36 + 16 + 4}$$

$$= \sqrt{56} \quad .$$

$$|PR| = \sqrt{((-2) - 4)^2 + (2 - 8)^2 + (4 - 0)^2} = \sqrt{4 + 36 + 16}$$

$$= \sqrt{56} \quad .$$

$$|QR| = \sqrt{((-2) - 2)^2 + (2 - 4)^2 + (-4 - (-2))^2} = \sqrt{16 + 4 + 36}$$

$$= \sqrt{56} \quad .$$

2. All the distances are the same ($\sqrt{56}$).
Ans.: It is an equilateral triangle since all the sides have equal length , namely: $\sqrt{56}$.

Problem Type 12.2b: Find an equation of the sphere with center $C(h,k,l)$ and radius r.

Example Problem 12.2b: Find an equation of the sphere with center $(1,2,-1)$ and radius 2.

Steps

1. Implement the formula

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

2. Expand everything and move everything to the left leaving 0 at the right side. Also rearange terms so that the quadratic terms come before the linear terms.

Example

1. In this problem $(h,k,l)=(1,2,-1)$ and $r=2$ so the equation is:

$$(x-1)^2+(y-2)^2+(z-(-1))^2=2^2 \quad ,$$

which is the same as

$$(x-1)^2+(y-2)^2+(z+1)^2=2^2 \quad .$$

2.

$$x^2-2x+1+y^2-4y+4+z^2+2z+1=4 \quad ,$$

Cleaning up:

$$x^2+y^2+z^2-2x-4y+2z+2=0 \quad .$$

Ans.: $x^2+y^2+z^2-2x-4y+2z+2=0$.

Problem Type 12.2c: Show that the equation represents a sphere, and find the center and radius.

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0 \quad .$$

Example Problem 12.2c: Show that the equation represents a sphere, and find the center and radius.

$$x^2 + y^2 + z^2 - 2x - 4y + 2z + 2 = 0 \quad .$$

Steps

1. The coefficients of x^2, y^2, z^2 should all be the same! If they are not, for example, if the equation is $x^2 + y^2 + 3z^2 + 2x + 6y - 5 + 11 = 0$ where the coefficients are not all the same, then it is **not** a sphere. Usually they are all 1, If the coefficient of x^2 (=coeff. of y^2=coeff. of z^2) is not 1, divide the whole equation by that coefficient, making it 1. The coeffs. of x^2, y^2, z^2 should now be *all* 1. Now group the terms so that x^2 is next to the x term, y^2 is next to the y term, and z^2 is next to the z term.

2. For each part separately, *complete the square*, using $X^2 + aX = (X + a/2)^2 - (a/2)^2$

Example

1. In this problem, the coeffs. of x^2 is already 1, as are those of y^2 and z^2. Grouping the x-terms, y-terms and z-terms, we get:

$$x^2 - 2x + y^2 - 4y + z^2 + 2z + 2 = 0$$

2.

$$x^2 - 2x = (x - 1)^2 - 1$$

$$y^2 - 4y = (y - 2)^2 - 4$$

$$z^2 + 2z = (z + 1)^2 - 1$$

So

$$x^2 - 2x + y^2 - 4y + z^2 + 2z + 2 = 0$$

becomes

$$(x-1)^2 - 1 + (y-2)^2 - 4 + (z+1)^2 - 1 + 2 = 0$$

3. Move all the numbers to the right and express the resulting number on the right as r^2. Compare with the equation of the sphere

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2 \quad ,$$

and read-off the **center** (h, k, l) and the **radious**, r.

3.

$$(x-1)^2 - 1 + (y-2)^2 - 4 + (z+1)^2 - 1 + 2 = 0$$

is the same as

$$(x-1)^2 + (y-2)^2 + (z+1)^2 = 4$$

which, in turn, is the same as

$$(x-1)^2 + (y-2)^2 + (z-(-1))^2 = 2^2 \quad ,$$

which is an equation of a sphere with **center** $(1, 2, -1)$ and **radius** 2.