Dr. Z's Math251 Handout #16.3 [The Fundamental Theorem for Line Integrals]

By Doron Zeilberger

Problem Type 16.3a: Determine whether or not

$$\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

is a conservative vector field. If it is, find a function f that $\mathbf{F} = \bigtriangledown f$.

Example Problem 16.3a: Determine whether or not

$$\mathbf{F} = (e^y + y\cos x)\mathbf{i} + (xe^y + \sin x + 2y)\mathbf{j}$$

is a conservative vector field. If it is, find a function f that $\mathbf{F} = \bigtriangledown f$.

Steps

1. Compute $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$. If they are **not** the same, then the vector field is **not** conservative. End of story. If they are, then it is a **conservative** vector field, and you must go on.

Example

1.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^y + y\cos x) = e^y + \cos x \quad .$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xe^y + \sin x + 2y) = e^y + \cos x \quad .$$

Since they are the same, the vector field is conservative. We must go on.

2. Look for a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = P \quad ,$$

$$\frac{\partial f}{\partial y} = Q \quad .$$

To this end, you integrate P w.r.t. to x

$$f = \int P\, dx \quad ,$$

getting an answer up to an **arbitrary constant** (from x's viewpoint), that is a function of y, $g(y)$. You then plug this tentative form of f into the second equation and find what $g'(y)$ is.

2. Since

$$\frac{\partial f}{\partial x} = e^y + y\cos x \quad ,$$

we get:

$$f = \int (e^y + y\cos x)\, dx = xe^y + y\sin x + g(y) \quad .$$

Plugging this to the second equation, we get

$$\frac{\partial}{\partial y}(xe^y + y\sin x + g(y)) = xe^y + \sin x + 2y \quad ,$$

which means

$$xe^y + \sin x + g'(y) = xe^y + \sin x + 2y$$

which means $g'(y) = 2y$.

3. Integrate the expression that you got for $g'(y)$ w.r.t. y, in order to get g, and plug it into f above.

3. $g(y) = \int 2y\, dy = y^2$ (now you don't have to bother about the C). Going back to f, we get

$$f = xe^y + y\sin x + y^2 \quad .$$

Ans.: $\mathbf{F}$ is conservative and the potential function f such that $\mathbf{F} = \nabla f$ is $f = xe^y + y\sin x + y^2$.

Problem Type 16.3b : (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k} \quad ,$$

$$C : x = x(t)\,,\ y = y(t)\,,\ z = z(t) \quad ; \quad a \le t \le b \quad .$$

Example Problem 16.3b: (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x,y,z) = (2xz + y^2)\,\mathbf{i} + 2xy\,\mathbf{j} + (x^2 + 3z^2)\,\mathbf{k} \quad ,$$

$$C : x = t^2 \quad , \quad y = t+1 \quad , \quad z = 2t-1 \quad , \quad 0 \le t \le 1 \quad .$$

Steps

1. We are looking for a function $f(x, y, z)$ such that $f_x = P$, $f_y = Q$, $f_z = R$. First integrate P w.r.t. to x getting

$$f = \int P \, dx = A(x, y, z) + g(y, z) \quad ,$$

where $A(x, y, z)$ is explicit but $g(y, z)$ is yet to be determined. then plug it into $f_y = Q$ getting

$$A_y + g_y = Q \quad ,$$

so $g_y = Q - A_y$, and integrating w.r.t. y gives

$$g(y, z) = \int (Q - A_y) \, dy = B(y, z) + h(z) \quad ,$$

where $B(y, z)$ is explicit but $h(z)$ is yet to be determined. Now we have

$$f = A(x, y, z) + B(y, z) + h(z) \quad ,$$

we plug it into $f_z = R$ and find $h'(z)$ from which we get h, and plug it back into f.

Example

1. $f_x = 2xz + y^2$ means

$$f = x^2 z + xy^2 + g(y, z) \quad .$$

$f_y = 2xy$ means

$$2xy + g_y = 2xy \quad ,$$

so $g_y = 0$ and $g(y, z) = h(z)$. So

$$f = x^2 z + xy^2 + h(z) \quad .$$

$f_z = x^2 + 3z^2$ means

$$x^2 + h'(z) = x^2 + 3z^2 \quad ,$$

so $h'(z) = 3z^2$ and

$$h(z) = z^3 \quad .$$

Going back to f we have

$$f = x^2 z + xy^2 + z^3 \quad .$$

2. Plug-in $t = a$, $t = b$ to get $\mathbf{r}(a)$ and $\mathbf{r}(b)$, and compute

$$f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \quad .$$

2.

$$\mathbf{r}(t) = \langle t^2, t+1, 2t-1 \rangle \quad ,$$

So

$$\mathbf{r}(0) = \langle 0, 1, -1 \rangle \quad ,$$

$$\mathbf{r}(1) = \langle 1, 2, 1 \rangle \quad .$$

Since our f equals $x^2z + xy^2 + z^3$, the value of the line integral is:

$$f(1,2,1) - f(0,1,-1) =$$

$$(1^2 \cdot 1 + 1 \cdot 2^2 + 1^3) - (0^2 \cdot (-1) + 0 \cdot 1^2 + (-1)^3) = 7 \quad .$$

Ans.: 7.