Dr. Z’s Math251 Handout #15.9 [Change of Variables in Multiple Integrals]

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**Problem Type 15.9a:** Find the Jacobian of the transformation

\[ x = g(u, v, w) \quad , \quad y = h(u, v, w) \quad , \quad z = k(u, v, w). \]

**Example Problem 15.9a:** Find the Jacobian of the transformation

\[ x = u^2v \quad , \quad y = v^2w \quad , \quad z = w^2u. \]

**Steps**

1. Compute all the entries in the Jacobian matrix

\[
\begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{vmatrix}.
\]

2. Evaluate the determinant:

\[
\begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{vmatrix} = (\frac{\partial x}{\partial u}) \begin{vmatrix}
\frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{vmatrix} - (\frac{\partial x}{\partial v}) \begin{vmatrix}
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial w}
\end{vmatrix} + (\frac{\partial x}{\partial w}) \begin{vmatrix}
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}
\end{vmatrix}.
\]

**Ans.:** \(9u^2v^2w^2.\)

**Problem Type 15.9b:** Use the given transformation to evaluate the integral

\[
\int \int_R F(x, y) \, dA,
\]

where \(R\) is the triangular region with vertices \((p_1, p_2), (q_1, q_2), (r_1, r_2)\); \(x = au + bv, y = cu + dv.\)
Example Problem 15.9b: Use the given transformation to evaluate the integral

\[ \int \int_R (x + y) \, dA \]

where \( R \) is the triangular region with vertices \((0, 0), (2, 1), (1, 2)\); \( x = 2u + v, \ y = u + 2v \).

Steps

1. Figure out the region in the \( uv \)-plane that gets transformed. Since a triangle goes to a triangle, we need to find the 3 vertices. Solve for \( u, v \) in terms of \( x, y \) and find the three points. Call the new triangle \( R' \).

   Since \( x = 2u + v, \ y = u + 2v \), when \((x, y) = (0, 0)\) \( u = 0, v = 0 \) so the point \((0, 0)\) goes to the point \((0, 0)\). When \((x, y) = (1, 2)\), we have to solve the system \( 1 = 2u + v, 2 = u + 2v \) giving us \( u = 0, v = 1 \) so \((1, 2)\) goes to \((0, 1)\). Similarly, \((2, 1)\) goes to \((1, 0)\). So the region in the \( uv \)-plane is the far simpler triangle whose vertices are \((0, 0), (1, 0), (0, 1)\). Let’s call this region \( R' \).

2. Find the Jacobian of the transformation. In this case of a so-called linear transformation, the Jacobian is simply \( ad - bc \). Also express \( F(x, y) \) in terms of \( (u, v) \) using the transformation.

   \[ \int \int_R F(x, y) \, dA = \int \int_{R'} F(au + bv, cu + dv)(ad - bc) \, dA \]

   The Jacobian is \( (2)(2) - (1)(1) = 3 \), so

   \[ \int \int_R (x + y) \, dA = \int \int_{R'} (2u + v + u + 2v) \cdot 3 \, dA = 9 \int \int_{R'} (u + v) \, dA \]
3. Draw the region (in this case triangle) in the $uv$-plane and express it as a type I (or type II) region. Then set-up the appropriate iterated integral, by deciding on the main road and the side streets.

3. The region is the triangle bounded by the axes and the line $u + v = 1$. It can be written as
\[ \{(u, v) | 0 \leq u \leq 1, 0 \leq v \leq 1 - u \} .\]

Our area-integral is thus equal to the iterated integral
\[ 9 \int_0^1 \int_0^{1-u} (u + v) \, dv \, du .\]

The inner integral is
\[ \int_0^{1-u} (u + v) \, dv = uv + \frac{v^2}{2} \bigg|_{0}^{1-u} = u(1 - u) + \frac{(1 - u)^2}{2} = \frac{(1 - u^2)}{2}, \]

and the whole integral is
\[ \frac{9}{2} \int_0^1 (1 - u^2) \, du = \frac{9}{2} \left[ u - \frac{u^3}{3} \right] \bigg|_{0}^{1} = \frac{9}{2} \cdot \frac{2}{3} = 3 .\]

Ans.: 3.