Dr. Z’s Math251 Handout #15.4 [Double Integrals in Polar Coordinates]

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**Problem Type 15.4a**: Evaluate the integral

\[ \int \int_D F(x, y) \, dA \, , \]

where \( D \) is a region best described in polar coordinates,

\[ D = \{ (r, \theta) \mid \alpha \leq \theta \leq \beta, \ h_1(\theta) \leq r \leq h_2(\theta) \} \ . \]

**Example Problem 15.4a**: Evaluate the integral

\[ \int \int_D e^{-x^2-y^2} \, dA \, , \]

where \( D \) is the region bounded by the semi-circle \( x = \sqrt{25-y^2} \) and the y-axis.

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**Steps**

1. Draw the region and express it, if possible and convenient, as

\[ D = \{ (r, \theta) \mid \alpha \leq \theta \leq \beta, \ h_1(\theta) \leq r \leq h_2(\theta) \} \ . \]

Of course, in many problems, the \( h_1(\theta) \) and/or \( h_2(\theta) \) may be plain numbers (i.e. not involve \( \theta \)).

**Example**

1. This is a semi-circle, i.e. half a circle, center origin, radius 5, and since it is bounded by the y-axis, and \( x \geq 0 \), it is the right half

[Had it been \( x = \sqrt{25-y^2} \) it would have been the left-half. Had it been \( y = \sqrt{25-x^2} \) it would have been the upper-half. Had it been \( y = -\sqrt{25-x^2} \) it would have been the lower-half.]

Since it is the right-half, \( \theta \) ranges from \( \theta = -\pi/2 \) (the downwards direction) to \( \theta = \pi/2 \) (the upwards direction). For each ray \( \theta = \theta_0 \), \( r \), the distance from the origin, ranges from \( r = 0 \) to \( r = 5 \) (and indeed does not depend on \( \theta \) in this problem). So our region phrased in polar coordinates is:

\[ D = \{ (r, \theta) \mid -\pi/2 \leq \theta \leq \pi/2, \ 0 \leq r \leq 5 \} \ . \]
2. Rewrite the area integral
\[ \int \int_D F(x, y) \, dA \]
in polar coordinates by replacing
\[ x \] by \( r \cos \theta \), \( y \) by \( r \sin \theta \), \( dA \) by \( r \, dr \, d\theta \).

[shortcut: Whenever you see \( x^2 + y^2 \) you can replace it by \( r^2 \).]

Write it as an iterated integral
\[ \int \int_D F(x, y) \, dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} F(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta , \]
with the \( \theta \)-integral being at the outside and the \( r \)-integral being in the inside.

3. Evaluate this iterated integral by first doing the inner-integral (possibly getting an expression in \( \theta \), or just a number), and then the outer integral.

3. The inside integral is (do the change-of-variable \( u = -r^2 \)):
\[ \int_0^5 e^{-r^2} \, r \, dr = (-1/2) e^{-r^2}\bigg|_0^5 = (1 - e^{-25})/2 \]
and the whole double-integral is
\[ \int_{-\pi/2}^{\pi/2} \int_0^5 e^{-r^2} \, r \, dr \, d\theta \]
\[ = \int_{-\pi/2}^{\pi/2} \left[ \int_0^5 e^{-r^2} \, r \, dr \right] \, d\theta \]
\[ = \int_{-\pi/2}^{\pi/2} \left[ (1 - e^{-25})/2 \right] \, d\theta = (1 - e^{-25})/2 \int_{-\pi/2}^{\pi/2} \, d\theta = \left[ (1 - e^{-25})/2 \right] \left[ \pi/2 - (-\pi/2) \right] = \pi(1 - e^{-25})/2 . \]
\text{Ans.:} \pi(1 - e^{-25})/2 .
Problem Type 15.4b: Find the volume of the solid above the surface \( z = f(x, y) \) and below the surface \( z = g(x, y) \).

Example Problem 15.4b: Find the volume of the solid above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \( x^2 + y^2 + z^2 = 2 \).

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Steps

1. Find the “floor”, let’s call it \( D \), by setting \( f(x, y) = g(x, y) \) (or if convenient already convert to polar coordinates).

2. The volume is the area integral of TOP-BOTTOM \[
\int \int_D [f(x, y) - g(x, y)] dA.
\]

Example

1. In polar coordinates, the two surfaces are \( z = r \) and \( z = \sqrt{2 - r^2} \). Setting them equal gives \( r = \sqrt{2 - r^2} \). Squaring both sides gives \( r^2 = 2 - r^2 \), which gives \( 2r^2 = 2 \), which gives \( r^2 = 1 \) and so \( r = \pm 1 \). But \( r \) is never negative, so \( r = -1 \) is nonsense. Hence the “floor”, \( D \), is the region bounded by the circle \( r = 1 \), or, if you wish, the disk \( r \leq 1 \).

So
\[
D = \{(r, \theta) | 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}.
\]

2. The bottom is \( z = \sqrt{x^2 + y^2} \), and in polar \( z = r \), and the top is \( x^2 + y^2 + z^2 = 2 \) which is \( z = \sqrt{2 - x^2 - y^2} \) and in polar \( z = \sqrt{2 - r^2} \). So the volume in polar coordinates is
\[
\int_0^{2\pi} \int_0^1 [\sqrt{2 - r^2} - r] r dr d\theta.
\]
3. Evaluate the iterated integral. First do the inner integral (w.r.t. to $r$) getting an expression in $\theta$ (or just a number), and then do the outer integral.

The inner integral is

$$
\int_0^1 \left[ \sqrt{2 - r^2} - r \right] r \, dr = \int_0^1 \left[ r \sqrt{2 - r^2} - r^2 \right] \, dr
$$

$$
= \int_0^1 r(2 - r^2)^{1/2} \, dr - \int_0^1 r^2 \, dr
$$

$$
= -(1/3)(2 - r^2)^{3/2}\bigg|_0^1 - r^3/3\bigg|_0^1
$$

$$
= -(1/3)(2 - r^2)^{3/2}\bigg|_0^1 - r^3/3\bigg|_0^1
$$

$$
= -(1/3)[(2 - 1^2)^{3/2} - (2 - 0^2)^{3/2}] - 1/3
$$

$$
= [2^{3/2} - 2]/3 = (2\sqrt{2} - 1)/3.
$$

The whole integral is thus:

$$
\int_0^{2\pi} \int_0^1 \left[ \sqrt{2 - r^2} - r \right] r \, dr \, d\theta
$$

$$
= \int_0^{2\pi} \left[ \int_0^1 \left[ \sqrt{2 - r^2} - r \right] r \, dr \right] \, d\theta
$$

$$
= \int_0^{2\pi} (2\sqrt{2} - 1)/3 \, d\theta
$$

$$
= 2\pi(2\sqrt{2} - 1)/3.
$$

**Ans.** The volume is $2\pi(2\sqrt{2} - 1)/3$.

**Problem Type 15.4c:** Evaluate the iterated integral by converting to polar coordinates.

$$
\int_a^b \int_{f_1(y)}^{f_2(y)} F(x, y) \, dx \, dy
$$

**Example Problem 15.4c:** Evaluate the iterated integral by converting to polar coordinates.

$$
\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 y \, dx \, dy
$$

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**Steps**

**Example**

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1. By looking at the limits of integration of the outer and inner integral signs, figure out the region $D$.

$$D = \{(x, y) \mid a \leq y \leq b, f_1(y) \leq x \leq f_2(y)\}$$

Draw this region, and express it in polar coordinates

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, g_1(\theta) \leq r \leq g_2(\theta)\}$$

2. Write the iterated integral as an area integral, then convert it to an iterated integral in polar coordinates. Use the “dictionary” $x = r \cos \theta$, $y = r \sin \theta$.

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 y \, dx \, dy$$

$$= \int_0^{\pi} \int_0^3 (r \cos \theta)^2 (r \sin \theta) r \, dr \, d\theta$$

$$= \int_0^{\pi} \int_0^3 r^4 \sin \theta \cos^2 \theta \, dr \, d\theta$$

1. Our region is:

$$D = \{(x, y) \mid 0 \leq y \leq 3, -\sqrt{9-y^2} \leq x \leq \sqrt{9-y^2}\}$$

Drawing it (do it!), we see that this is the upper-half of the circle whose center is the origin and whose radius is 3. In polar coordinates it is:

$$D = \{(r, \theta) \mid 0 \leq \theta \leq \pi, 0 \leq r \leq 3\}$$
3. Evaluate that iterated integral by doing the inner integral first, and then the outer integral.

The inner integral is:

\[
\int_0^3 r^4 \sin \theta \cos^2 \theta \, dr = \sin \theta \cos^2 \theta \int_0^3 r^4 \, dr
\]

\[
= \sin \theta \cos^2 \theta \left[ \frac{r^5}{5} \right]_0^3
\]

\[
= \frac{243}{5} \sin \theta \cos^2 \theta .
\]

The outer integral is:

\[
\int_0^\pi \left( \int_0^3 r^4 \sin \theta \cos^2 \theta \, dr \right) \, d\theta
\]

\[
= \int_0^\pi \left( \frac{243}{5} \cos^2 \theta \sin \theta \right) \, d\theta
\]

\[
= \frac{243}{5} \left[ -\cos^3 \theta \right]_0^\pi = \frac{81}{5} \cdot (-\cos^3(\pi) - (-\cos^3(0))) = \frac{162}{5} .
\]

Ans.: \(\frac{162}{5}\).