Problem Type 14.8a: Use Lagrange multipliers to find the maximum and minimum values of
the function subject to the given conditions.

\[ f(x, y, z) = \text{Expression}(x, y, z) \ , \ g(x, y, z) = k \ . \]

Example Problem 14.8a: Use Lagrange multipliers to find the maximum and minimum values
of the function subject to the given conditions.

\[ f(x, y, z) = xyz \ ; \ 3x^2 + 2y^2 + z^2 = 6 \ . \]

Steps

1. Find the gradients of \( f(x, y, z) \) and \( g(x, y, z) \).

Example

1. \( f_x = \frac{\partial}{\partial x} xyz = yz \), \( f_y = \frac{\partial}{\partial y} xyz = xz \),
   \( f_z = \frac{\partial}{\partial z} xyz = xy \). So,

   \[ \nabla f = (yz, xz, xy) \ . \]

   \[ g_x = \frac{\partial}{\partial x}(3x^2 + 2y^2 + z^2) = 6x \ , \ g_y = \frac{\partial}{\partial y}(3x^2 + 2y^2 + z^2) = 4y \ , \ g_z = \frac{\partial}{\partial z}(3x^2 + 2y^2 + z^2) = 2z \). So

   \[ \nabla g = (6x, 4y, 2z) \ . \]

2. Introduce another variable, \( \lambda \), and set up the equations implied by

\[ \nabla f = \lambda \nabla g \ . \]

To them add, the equation \( g(x, y, z) = k \).

2. \( \nabla f = \lambda \nabla g \) means

\[ (yz, xz, xy) = \lambda(6x, 4y, 2z) \ , \]

that spells-out to the set of equations

\[ yz = 6\lambda x \ , \ xz = 4\lambda y \ , \]

\[ xy = 2\lambda z \ , \ 3x^2 + 2y^2 + z^2 = 6 \ . \]
3. Use algebra to solve the system of four equations and four unknowns.

3. Multiplying the first three equations, we get \((xyz)^2 = 48\lambda^3\) so \(xyz = 48\lambda^3\), and hence \(yz = 48\lambda^3/x\) and we get from the first equation \(48\lambda^3/x = 6\lambda\). This means \(8\lambda^2 = x^2\) and so \(x = \sqrt{8}\lambda\).

From \(xyz = 48\lambda^3\), we also get \(xz = 48\lambda^3/y\) and we get from the second equation \(48\lambda^3/y = 4\lambda y\). This means \(12\lambda^2 = y^2\) and so \(y = \sqrt{12}\lambda\).

From \(xyz = 48\lambda^3\), we also get \(xy = 48\lambda^3/z\) and we get from the third equation \(48\lambda^3/z = 2\lambda z\). This means \(24\lambda^2 = z^2\) and so \(z = \sqrt{24}\lambda\).

Plugging these expressions in \(\lambda\) into the last equation \(3x^2 + 2y^2 + z^2 = 6\) we get

\[3(\sqrt{8}\lambda)^2 + 2(\sqrt{12}\lambda)^2 + (\sqrt{24}\lambda)^2 = 6\]

So,

\[(3 \cdot 8 + 2 \cdot 12 + 24)\lambda^2 = 6\]

\[\lambda^2 = \frac{1}{12}\]

and so

\[\lambda = \pm \frac{1}{\sqrt{12}}\]

We get two solutions. The first one is

\[\lambda = \frac{1}{\sqrt{12}}\]

\[x = \sqrt{8} \cdot \frac{1}{\sqrt{12}} = \sqrt{2}/3\]

\[y = \sqrt{12} \cdot \frac{1}{\sqrt{12}} = 1\]

\[z = \sqrt{24} \cdot \frac{1}{\sqrt{12}} = \sqrt{2}\]

which means the point \((\sqrt{2}/3, 1, \sqrt{2})\).

And the second is

\[\lambda = -\frac{1}{\sqrt{12}}\]

\[x = \sqrt{8} \cdot \frac{1}{\sqrt{12}} = -\sqrt{2}/3\]

\[y = \sqrt{12} \cdot \frac{-1}{\sqrt{12}} = -1\]

\[z = \sqrt{24} \cdot \frac{-1}{\sqrt{12}} = -\sqrt{2}\]

which means the point \((-\sqrt{2}/3, -1, -\sqrt{2})\).
4. Now you can forget about the $\lambda$ and plug-in these point(s) into $f$ and see who gives the largest value, that is the maximum value and who is the smallest, that is the minimum value.

\[ f(\sqrt{2/3}, 1, \sqrt{2}) = \frac{2}{\sqrt{3}} \]
\[ f(-\sqrt{2/3}, -1, -\sqrt{2}) = -\frac{2}{\sqrt{3}} \]

Ans.: The maximum value is $2/\sqrt{3}$ and the minimum value is $-2/\sqrt{3}$. 