Dr. Z's Math251 Handout #14.6 [Directional Derivatives and the Gradient Vector]

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Problem Type 14.6a: Find the directinal derivative of the function $f(x, y, z)$ at the point (x_0, y_0, z_0) in the direction $\langle v_1, v_2, v_3 \rangle$.

Example Problem 14,6a: Find the directinal derivative of the function $f(x, y, z) = \ln(x^2+y^2+z^2)$ at the point $(2, 1, 3)$ in the direction $\langle 1, 2, 2 \rangle$.

Steps

1. Find the **gradient** $\nabla f = \langle f_x, f_y, f_z \rangle$ by taking all the first partial derivatives. Also find the unit vector in the direction of $\langle v_1, v_2, v_3 \rangle$ by dividing by its length.

2. Plug-in $x = x_0, y = y_0, z = z_0$ into ∇f .

Example

1.

$$f_x = \frac{2x}{x^2 + y^2 + z^2} \ , \ f_y = \frac{2y}{x^2 + y^2 + z^2} \ , \ f_z = \frac{2z}{x^2 + y^2 + z^2} \quad .$$

So

$$\nabla f = \langle \frac{2x}{x^2 + y^2 + z^2}, \frac{2y}{x^2 + y^2 + z^2}, \frac{2z}{x^2 + y^2 + z^2} \rangle$$

$|\langle 1, 2, 2 \rangle| = \sqrt{1^2 + 2^2 + 2^2} = 3$, so

$$\mathbf{u} = \frac{1}{3}\langle 1, 2, 2 \rangle = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle \quad .$$

2.

$$\nabla f(2, 1, 3) = \langle \frac{2 \cdot 2}{2^2 + 1^2 + 3^2}, \frac{2 \cdot 1}{2^2 + 1^2 + 3^2}, \frac{2 \cdot 3}{2^2 + 1^2 + 3^2} \rangle$$

$$= \langle \frac{2}{7}, \frac{1}{7}, \frac{3}{7} \rangle \quad .$$

3. Take the dot product $\bigtriangledown f.\mathbf{u}$.

3.
$$\langle\frac{2}{7},\frac{1}{7},\frac{3}{7}\rangle.\langle\frac{1}{3},\frac{2}{3},\frac{2}{3}\rangle$$
$$=\frac{2}{7}\cdot\frac{1}{3}+\frac{1}{7}\cdot\frac{2}{3}+\frac{3}{7}\cdot\frac{2}{3}=\frac{10}{21}$$

Ans.: The requested directional derivative is $\frac{10}{21}$.

Problem Type 14.6b: Find the maximum rate of change of f at the given point and the direction in which it occurs.

$$f(x,y)=Expression(x,y) \quad , \quad (x_0,y_0) \quad .$$

Example Problem 14.6b: Find the maximum rate of change of f at the given point and the direction in which it occurs.

$$f(x,y)=\sin(xy) \quad , \quad (1,0).$$

Steps

Example

1. Find the gradient

$$\bigtriangledown f=\langle f_x,f_y\rangle$$

1. $f_x=y\cos(xy), f_y=x\cos(xy)$. So

$$\bigtriangledown f=\langle f_x,f_y\rangle=\langle y\cos(xy),x\cos(xy)\rangle \quad .$$

2. Plug-in $x=x_0, y=y_0$ into $\bigtriangledown f$.

2.

$$\bigtriangledown f(1,0)=\langle 0\cdot\cos(0),1\cdot\cos(0)\rangle=\langle 0,1\rangle \quad .$$

3. The maximum rate of change of f is simply the length of $\bigtriangledown f$ at the designated point. The direction in which is occurs is that direction. So find the unit vector in that direction.

3.
$$|\langle 0,1\rangle|=\sqrt{0^2+1^2}=1 \quad .$$

$\langle 0,1\rangle$ is already a unit vector, so the direction is $\langle 0,1\rangle$.

Ans.: The maximum rate of change is 1 in the direction $\langle 0,1\rangle$ (or $\mathbf{j}$).