Dr. Z's Math251 Handout #14.3 [Partial Derivatives]

By Doron Zeilberger

Problem Type 14.3a: Find the first partial derivatives of the function $f(x, y)$.

Example Problem 14.3a: Find the first partial derivatives of the function

$$f(x, y) = x^3 y + \ln(x + y) \quad .$$

Steps

1. When you take the partial derivative with respect to x, then x is the boss!, and y is to be treated just like a constant (number) (it may even help, in your mind to replace y by y_0 to bring home the fact that it is to be treated as a mere constant, then at the end replace y_0 by y).

2. When you take the partial derivative w.r.t. to y it is the opposite: y is the boss and x is treated as a constant.

Example

1.
$$\frac{\partial f}{\partial x} = (x^3 y + \ln(x + y))' \quad ,$$

where at the *present context* the prime $'$ means differentiation with respect to x. So this equals

$$3x^2 y + \frac{1}{x + y} \quad .$$

2.
$$\frac{\partial f}{\partial y} = (x^3 y + \ln(x + y))' \quad ,$$

where at the *present context* the prime $'$ means differentiation with respect to y. So this equals

$$x^3 + \frac{1}{x + y} \quad .$$

Ans.: $\frac{\partial f}{\partial x} = 3x^2 y + \frac{1}{x+y}$, $\frac{\partial f}{\partial y} = x^3 + \frac{1}{x+y}$

.

Problem Type 14.3b: Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, given the relationship

$$LeftSide(x, y, z) \; = \; RightSide(x, y, z) \quad .$$

Example Problem 14.3b: Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, given the relationship

$$x^2 + y^3 + z^2 = 4xy^2z \quad .$$

Steps **Example**

1. When you take the partial derivative with respect to x, then x is the boss!, and y is to be treated just like a constant (number). Also right now, z is to be treated as a function of x!

1. Differentiate with respect to x the given relation:

$$(x^2 + y^3 + z^2)' = (4xy^2 z)' \quad ,$$

where at the *present context* the prime $'$ means differentiation with respect to x. So this becomes

$$2x + 0 + 2zz' = (4y^2)[xz]' \quad ,$$

where we have taken $4y^2$ out since, in the present context, it is a **constant**. By the product rule applied to $[xz]'$,

$$2x+2zz' = (4y^2)[x'z+xz'] = (4y^2)[z+xz'] \quad .$$

So much for calculus. We now have to use **algebra** to solve for z' (i.e. $\frac{\partial z}{\partial x}$). Opening up parentheses,

$$2x + 2zz' = 4y^2 z + 4y^2 xz' \quad .$$

Moving all the terms involving z' to the left and all the other terms to the right yields

$$2zz' - 4y^2 xz' = 4y^2 z - 2x \quad .$$

Factoring z' out,

$$(2z - 4y^2 x)z' = 4y^2 z - 2x \quad .$$

and finally, dividing by the term in front of z', to get z' to be on its own, we get:

$$z' = \frac{4y^2 z - 2x}{2z - 4y^2 x} = \frac{2y^2 z - x}{z - 2y^2 x} \quad .$$

This is $\frac{\partial z}{\partial x}$.

2. When you take the partial derivative with respect to y, then y is the boss!, and x is to be treated just like a constant (number). *Now* z is to be treated as a function of y!

2. Differentiate with respect to y the given relation gives:

$$(x^2 + y^3 + z^2)' = (4xy^2z)'$$

where at the *present context* the prime $'$ means differentiation with respect to y. So this becomes

$$0 + 3y^2 + 2zz' = (4x)[y^2z]' \quad ,$$

where we have taken $4x$ out since, in the present context, it is a **constant**. By the product rule applied to $[y^2z]'$,

$$3y^2 + 2zz' = (4x)[(y^2)'z + y^2z']$$

So

$$3y^2+2zz' = (4x)[2yz+y^2z'] = 8xyz+4xy^2z' \quad .$$

So much for calculus. We have now to use **algebra** to solve for z' (i.e. $\frac{\partial z}{\partial y}$). Moving all the terms involving z' to the left and all the other terms to the right yields

$$2zz' - 4xy^2z' = 8xyz - 3y^2 \quad .$$

Factoring z' out,

$$(2z - 4xy^2)z' = 8xyz - 3y^2 \quad ,$$

and finally, dividing by the term in front of z', to get z' to be on its own, we get:

$$z' = \frac{8xyz - 3y^2}{2z - 4xy^2} = \frac{(8xz - 3y)y}{2(z - 2xy^2)} \quad .$$

This is $\frac{\partial z}{\partial y}$.