Problem Type 13.3a: Find the length of the curve \( r(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \), \( t_0 \leq t \leq t_1 \).

Example Problem 13.3a: Find the length of the curve \( r(t) = t^2 \mathbf{i} + 2t \mathbf{j} + \ln t \mathbf{k} \), \( 1 \leq t \leq e \).

Steps

1. Compute the derivative
   \[
   r'(t) = x'(t) \mathbf{i} + y'(t) \mathbf{j} + z'(t) \mathbf{k} \, .
   \]
   \[
   r'(t) = (t^2)' \mathbf{i} + (2t)' \mathbf{j} + (\ln t)' \mathbf{k} = 2t \mathbf{i} + 2 \mathbf{j} + \frac{1}{t} \mathbf{k} \, .
   \]

2. Find the magnitude of \( r'(t) \),
   \[
   |r'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, ,
   \]
   and use algebra and/or trig to simplify as much as you can.
   \[
   |r'(t)| = \sqrt{(2t^2 + 2^2 + \frac{1}{t^2})} = \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \sqrt{\frac{(2t^2 + 1)^2}{t^2}} = \frac{(2t^2 + 1)}{t} = 2t + \frac{1}{t} \, .
   \]

3. Integrate the expression that you got in step 2 from \( t_0 \) to \( t_1 \).
   \[
   \int_{t_0}^{t_1} |r'(t)| \, dt
   \]
   \[
   = \int_{1}^{e} |r'(t)| \, dt = \int_{1}^{e} \left[2t + \frac{1}{t}\right] \, dt = e^2 + \ln e - \left(1^2 + \ln 1\right) = e^2 + 1 - (1 + 0) = e^2 \, .
   \]
   Ans.: The arc length of that curve is \( e^2 \).

Problem Type 13.3b: Reparametrize the curve with respect to arc length measured from the point when \( t = t_0 \) in the direction of increasing \( t \).

Example Problem 13.3b: Reparametrize the curve with respect to arc length measured from
the point when $t = 0$ in the direction of increasing $t$.

$$r(t) = 5 \sin t \, \mathbf{i} + 3 \, \mathbf{j} + 5 \cos t \, \mathbf{k}$$

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**Steps**

1. Compute $r'(t)$, and then take its magnitude $|r'(t)|$.

   
   $r'(t) = (5 \sin t)' \, \mathbf{i} + 3' \, \mathbf{j} + (5 \cos t)' \, \mathbf{k}$

   $= (5 \cos t) \, \mathbf{i} - (5 \sin t) \, \mathbf{k}$

   So

   
   $$|r'(t)| = \sqrt{(5 \cos t)^2 + (-5 \sin t)^2} = \sqrt{25(\cos^2 t + \sin^2 t)} = 5$$

2. Integrate it from $t_0$ to $t_1$. Get an expression in terms of $t_1$ and call it $s$. Now change $t_1$ into $t$. Now solve for $t$ in terms of $s$.

   
   $$s = \int_{t_0}^{t_1} 5 \, dt = 5t_1$$

   Changing the $t_1$ into $t$ we get

   $$s = 5t$$

   and expressing $t$ in terms of $s$, we get

   $$t = s/5$$

3. Go back to the original $r(t)$ and replace $t$ by the expression in $s$ that you found in step 2.

   
   $$r(t) = 5 \sin t \, \mathbf{i} + 3 \, \mathbf{j} + 5 \cos t \, \mathbf{k}$$

   becomes

   $$5 \sin(s/5) \, \mathbf{i} + 3 \, \mathbf{j} + 5 \cos(s/5) \, \mathbf{k}$$

   This is the **Ans.**
**Problem Type 13.3c:** Find the curvature for

\[ \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} . \]

**Example Problem 13.3c:** Find the curvature for

\[ \mathbf{r}(t) = t \mathbf{i} + 2t \mathbf{j} + t^2 \mathbf{k} . \]

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**Steps**

1. Compute \( \mathbf{r}'(t) \) and \( \mathbf{r}''(t) \).
   
   \( \mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 2t \mathbf{k} \).
   
   \( \mathbf{r}''(t) = 2 \mathbf{k} \).

2. Compute the cross product \( \mathbf{r}'(t) \times \mathbf{r}''(t) \).
   
   \[
   \begin{vmatrix}
   \mathbf{i} & \mathbf{j} & \mathbf{k} \\
   1 & 2 & 2t \\
   0 & 0 & 2 \\
   \end{vmatrix}
   = \begin{vmatrix}
   2 & 2t \\
   0 & 2 \\
   \end{vmatrix} \mathbf{i} - \begin{vmatrix}
   1 & 2t \\
   0 & 2 \\
   \end{vmatrix} \mathbf{j} + \begin{vmatrix}
   1 & 2 \\
   0 & 0 \\
   \end{vmatrix} \mathbf{k}
   = 4\mathbf{i} - 2\mathbf{j}
   
   \]

3. Find the magnitude of the vector that you found in step 2 (namely \( \mathbf{r}'(t) \times \mathbf{r}''(t) \)).
   
   Also find the magnitude of \( \mathbf{r}'(t) \), and finally use the formula for the curvature
   
   \[ \kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \]
   
   \[
   |\mathbf{r}'(t)| = |\mathbf{i} + 2\mathbf{j} + 2t \mathbf{k}| \\
   = \sqrt{1^2 + 2^2 + (2t)^2} = \sqrt{5 + 4t^2}
   
   Finally,
   
   \[
   \kappa(t) = \frac{\sqrt{20}}{(\sqrt{5 + 4t^2})^3}
   
   This is the \textbf{Ans..}