**Solutions to MATH 152 (07-09), Dr. Z. , First Midterm, Thurs. Oct. 18, 2012.**

**1**. (10 points [5 each]) Evaluate the following definite integrals:
**(a)**

$$\int_0^1 xe^{-2x}\,dx \quad ;$$

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**Ans. to (a):** $-\frac{3}{4}e^{-2}+\frac{1}{4}$ .

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We first evaluate the *indefinite integral*, using **integration by parts**. $u=x$, $v'=e^{-2x}$ so $u'=1$ and $v=-\frac{1}{2}e^{-2x}$, and we have

$$\int xe^{-2x}\,dx = x(-\frac{1}{2}e^{-2x}) - \int (1)(-\frac{1}{2}e^{-2x})\,dx = x(-\frac{1}{2}e^{-2x}) + \frac{1}{2}\int e^{-2x}\,dx$$

$$= x(-\frac{1}{2}e^{-2x}) + \frac{1}{2}\frac{e^{-2x}}{-2}\,dx = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$$

We now compute the **definite** integral

$$\int_0^1 xe^{-2x}\,dx \quad = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\Big|_0^1 = (-\frac{1}{2}e^{-2}) - \frac{1}{4}e^{-2}) - (-\frac{0}{2}e^{-2x}) - \frac{1}{4}e^0 = -\frac{3}{4}e^{-2} + \frac{1}{4} \quad .$$

**Comment**: The answer is of **type number**.
**(b)**

$$\int_2^3 \frac{3x^2-1}{x^3-x}\,dx \quad .$$

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**Ans. to (b):** $\ln 4$ .

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I told you many times that before you try anything fancy or complicated, the first thing to do is see whether you are lucky and the TOP happens to be the derivative of BOT. In that lucky case the answer is immediate (for the indefinite integral): $\ln|BOT| + C$.
Since $(x^3 - x)' = 3x^2 - 1$, we have *immediately*

$$\int \frac{3x^2 - 1}{x^3 - x}\, dx = \ln|x^3 - x| + C \quad ,$$

so the definite integral is

$$\int_2^3 \frac{3x^2 - 1}{x^3 - x}\, dx = \ln|3^3 - 3| - \ln|2^3 - 2| = \ln|24| - \ln|6| = \ln 24 - \ln 6 = \ln(24/6) = \ln 4 \quad .$$

**Comments:** 1. The answer is of **type number** (being a definite integral). 2. Most people tried to do it the long way, using partial fractions, and some even got a correct answer (after wasting many valuable minutes) but most of those people made computational errors and didn't get the right answer (if they did it the correct (albeit long) way I gave partial credit). 3. Amongst those people who got it right, most people left it as $\ln|24| - \ln|6|$ or, at best, $\ln|4|$. This is correct, but shame on you. **The absolute value of 4 is 4**!

**2**. (10 points ) The base of a solid is the region inside the circle $x^2 + y^2 = 9$. Each cross section of the solid perpendicular to the $y$-axis is a square. What is the volume of the solid?

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**Ans.** 144 square-units.

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The area of a square is $(side)^2$. The length of a side perpendicular to the $y$-axis at the point $(0, y)$ is $2x$ so the area of a cross-section is $A(y) = (2x)^2 = 4x^2$. Using the defining equation of our curve $x^2 + y^2 = 9$ we can write it in terms of $x$ (as we must) and we get

$$A(y) = 4(9 - y^2) \quad .$$

The extent of the circle is from $y = -3$ to $y = 3$ (it is a circle of radius 3), so the *limits of integration* are $-3$ and 3 and we have

$$Volume = \int_{-3}^{3} A(y)\, dy = \int_{-3}^{3} 4(9 - y^2)\, dy = 4(9y - \frac{y^3}{3})\Big|_{-3}^{3}$$

$$4(9 \cdot 3 - \frac{3^3}{3}) - 4(9 \cdot (-3) - \frac{(-3)^3}{3}) = 4 \cdot 18 - 4 \cdot (-18) = 72 + 72 = 144 \quad .$$

**3**. (10 points, 5 each) Consider the region lying above the $x$-axis, below the line $y = x$ and between the vertical lines $x = 0$ and $x = 1$. Find the volume formed by rotating it about

(a) the horizontal line $y = -1$

(b) the vertical line $x = -1$ .

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**Ans. to (a):** $\frac{4\pi}{3}$ **Ans. to (b):** $\frac{5\pi}{3}$

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**(a)**: Recall that by the method of washers (where the axis of rotation is parallel to the $x$-axis, of the form $y = c$ and it is below the region)

$$Volume = \pi \int_a^b ((TOP - c)^2 - (BOT - c)^2)\, dx \quad .$$

In our region $a = 0, b = 1$, $BOT = 0$ (the $x$-axis is the line $y = 0$) and $TOP = x$, and $c = -1$, so

$$Volume = \pi \int_0^1 ((x-(-1))^2-(0-(-1))^2)\, dx = \pi \int_0^1 ((x+1)^2-1)\, dx = \pi \int_0^1 (x^2+2x)\, dx =$$

$$\pi \left(\frac{x^3}{3} + x^2\right)\Big|_0^1 = \pi \left(\frac{1^3}{3} + 1^2 - 0\right) = \pi\frac{4}{3} = \frac{4\pi}{3} \quad .$$

**(b)**: By the method of *cylindrical shells* the if region above the $x$-axis and below the graph of $y = f(x)$, between the vertical lines $x = a$ and $x = b$ is rotated around the vertical line $x = c$ (that lies to the left of the region)

$$Volume = 2\pi \int_a^b (x - c) f(x)\, dx$$

Here $a = -1, b = 1, c = -1$ and $f(x) = x$ and we get

$$Volume = 2\pi \int_0^1 (x - (-1))x\, dx = 2\pi \int_0^1 (x + 1)x\, dx = 2\pi \int_0^1 (x^2 + x)\, dx$$

$$= 2\pi(\frac{x^3}{3} + \frac{x^2}{2})\Big|_0^1 = 2\pi(\frac{1^3}{3} + \frac{1^2}{2}) = 2\pi(\frac{1}{3} + \frac{1}{2}) = 2\pi(\frac{5}{6}) = \frac{5\pi}{3} \quad .$$

**4**. (10 pts) Find the area bounded between the curve $y = 2x^3 + 1$ and the line $y = 2x + 1$ .

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**Ans.** 1 square units.

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We first find for what values of $x$ the curves meet each other, by setting them equal to each other, and solving for $x$:

$$2x^3 + 1 = 2x + 1 \quad ,$$

means

$$2x^3 - 2x = 0 \quad .$$

Factorizing:

$$2(x+1)x(x-1) = 0 \quad ,$$

yielding the **three** locations $x = -1, x = 0, x = 1$. So we have to consider separately the interval from $x = -1$ to $x = 0$ and the interval from $x = 0$ to $x = 1$.
In the first interval, plugging-in $x = -1/2$ into $y = 2x^3 + 1$ gives $\frac{3}{4}$ and into $y = 2x + 1$ gives 0 so this means that $TOP = 2x^3 + 1, BOT = 2x + 1$ there.
In the second interval, plugging-in $x = 1/2$ into $y = 2x^3 + 1$ gives $\frac{5}{4}$ and into $y = 2x + 1$ gives 2 so this means that $TOP = 2x + 1, BOT = 2x^3 + 1$ there.
So the total area is

$$\int_{-1}^{0} [(2x^3+1)-(2x+1)]\,dx + \int_{0}^{1} [(2x+1)-(2x^3+1)]\,dx = \int_{-1}^{0} (2x^3-2x)\,dx + \int_{0}^{1} (2x-2x^3)\,dx$$

$$= 2\int_{-1}^{0} (x^3 - x)]\,dx + 2\int_{0}^{1} (x - x^3)\,dx$$

$$= 2(\frac{x^4}{4} - \frac{x^2}{2})\Big|_{-1}^{0} + 2(\frac{x^2}{2} - \frac{x^4}{4})\Big|_{0}^{1}$$

$$= 2(0 - (\frac{(-1)^4}{4} - \frac{(-1)^2}{2})) \; + \; 2(\frac{1^2}{2} - \frac{1^4}{4} - 0) = 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = \frac{1}{2} + \frac{1}{2} = 1 \quad .$$

**5**. (10 points, 5 each) Determine whether each of the following improper integrals is convergent or divergent. Evaluate those that are convergent, if possible. Be sure to explain everything.

**(a)**

$$\int_0^4 \frac{1}{\sqrt{x}}\, dx$$

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**Ans to (a).** convergent, its value is 4.

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Using the **short way** we have

$$\int_0^4 \frac{1}{\sqrt{x}} = \int_0^4 x^{-\frac{1}{2}} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Big|_0^4 = 2\sqrt{x}\Big|_0^4 = 2(\sqrt{4}-0) = 2\cdot 2 = 4 \quad .$$

Since we got a **finite number** the improper integral is convergent and its value is 4.
Using the **long (official) way** we have

$$\int_0^4 \frac{1}{\sqrt{x}} = \lim_{c\to 0}\int_c^4 \frac{1}{\sqrt{x}} = \lim_{c\to 0}\int_c^4 x^{-\frac{1}{2}} = \lim_{c\to 0}\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Big|_c^4 = \lim_{c\to 0} 2\sqrt{x}\Big|_c^4 = \lim_{c\to 0} 2(\sqrt{4}-\sqrt{c}) = 2\cdot(2-0) = 4 \quad .$$

Since we got a **finite number** the improper integral is convergent and its value is 4.

**(b)**

$$\int_{10}^{\infty} \frac{x^{199}+x^{76}+1}{x^{201}-x^{76}+7}\, dx$$

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**Ans to (b).** convergent; impossible fo do without computer (and even a compter would only give an approximation)

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By the **limit comparison rule**, the convergence vs. divergence status of this extremely complicated integral is the **same** as of the much simpler

$$\int_{10}^{\infty} \frac{x^{199}}{x^{201}}\, dx$$

obtained by **only** retaining the **leading terms**. Using algebra, this equals

$$\int_{10}^{\infty} \frac{1}{x^2}\, dx \quad .$$

By the $p$-test, since $p = 2$ is larger than 1 this new integral is convergent. So the answer is "convergent by combining the limit-comparison test with the $p$-test".

**6**. (10 pts) Find the average value of the function $f(x) = x^2$ on the interval $0 \leq x \leq 3$. Is it larger or smaller than the average of the maximum and minimum of $x^2$ on that interval?

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**Answers:** Average=3 Max=9 Min=0 ; smaller

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$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3-0}\int_0^3 x^2\,dx$$

$$= \frac{1}{3}\frac{x^3}{3}\Big|_0^3 = \frac{1}{9}(3^3 - 0^3) = \frac{1}{9}(27) = 3 \quad .$$

Since $f(x) = x^2$ is obviously **increasing** (i.e. it goes **uphill**, when moving from left to right) its minimum value is at the beginning, $x = 0$ so it equals $0^2 = 0$ and its maximum value is at the end, $x = 3$ so it equals $3^2 = 9$. The simple average of two numbers $A$ and $B$ is $(A+B)/2$ so the average of the max and min values is $(9+0)/2 = 4.5$ So the average of the function over the interval $[0, 3]$ being 3 is **smaller** than the average of its extreme values.

**7**. (10 pts [6 for (a) and 4 for (b)]) Let

$$I = \int_0^4 x^2\, dx \quad ;$$

(**Reminders**:

$$T_N = \frac{1}{2}\Delta x\,[y_0 + 2y_1 + \ldots + 2y_{N-1} + y_N] \quad ,$$

where $\Delta x = \frac{b-a}{N}$, and $y_j = f(a + j\Delta x)$. Also recall

$$Error(T_N) \le \frac{K_2(b-a)^3}{12N^2} \quad ,$$

where $K_2$ is a number that that $|f''(x)| \le K_2$ for all $x \in [a, b]$.)

(a) Use The trapezoid rule with $N = 4$ subdivisions to find an approximation, call it $J$.

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**Ans to (a)** 22

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First $\Delta x = (4-0)/4 = 1$. We have:

$$x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4 \quad ,$$

so

$$y_0 = 0^2 = 0, y_1 = 1^2 = 1, y_2 = 2^2 = 4, y_3 = 3^2 = 9, y_4 = 4^2 = 16 \quad .$$

So

$$T_4 = \frac{1}{2}[1\cdot 0 + 2\cdot 1 + 2\cdot 4 + 2\cdot 9 + 1\cdot 16] = \frac{1}{2}[0 + 2 + 8 + 18 + 16] \;=\; \frac{1}{2}[44] = 22 \quad .$$

(b) Use the error estimate to find an upper bound for the error $|I - J|$.

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**Ans to (b)** $\frac{2}{3}$

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since $f'(x) = 2x$ we have $f''(x) = 2$, a constant!, so the *min value* of its absolute value is 2. So $K_2 = 2$.

$$Error(T_4) \le \frac{K_2(b-a)^3}{12N^2} = \frac{2(4-0)^3}{12\cdot 4^2} = \frac{2\cdot 4}{12} = \frac{8}{12} = \frac{2}{3} \quad .$$

**8**. (10 points, 5 each) (a) Evaluate

$$\int \sec^4 x\, dx \quad .$$

[Reminder: $\sec^2 x - \tan^2 x = 1$]

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**Ans to (a)** $\tan x + \frac{1}{3}\tan^3 x + C$

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Since we were not given a *reduction formula*, we must use *trig. substitution*. Luckily the power of $\sec x$ is **even**, so this method is applicable (remember sec *loves* even, tan *loves* odd, sin and cos both *love* odd). Whenever we use trig. substitution we always substitute $u = TheOtherrGuy$. For $\sec x$ the "other guy" is $\tan x$ so

$$u = \tan x$$

Next

$$\frac{du}{dx} = \sec^2 x \quad so \quad dx = \frac{du}{\sec^2 x} \quad .$$

So

$$\int \sec^4 x\, dx = \int (\sec^4 x)\frac{du}{\sec^2 x} = \int \sec^2 x\, du \quad .$$

But thanks to the trig. identity $\sec^2 x = \tan^2 x + 1$ this equals

$$\int (1 + u^2)\, du \quad .$$

This is a calc1 simple integral that equals $u + \frac{u^3}{3}$. Going back to the $x$-language we get

$$\tan x + \frac{1}{3}\tan^3 x + C \quad .$$

(b) Evaluate

$$\int \left(3\ln x + x + \sin^2 x\right)\, dx \quad .$$

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**Ans to (b)** $3x(\ln x - 1) + \frac{x^2}{2} + \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

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First we break it up into its constituents:

$$\int \left(3\ln x + x + \sin^2 x\right)\, dx = 3\int \ln x\, dx + \int x\, dx + \int \sin^2 x\, dx \quad .$$

Now we have to tackle three **subproblems**.

$$\int \ln x\, dx \quad ,$$

you do by integration-by-parts: $u = \ln x, v' = 1$ so $u' = 1/x, v = x$ and $\int \ln x\, dx = x\ln x - \int (1/x)(x)\, dx = x\ln x - \int (1)\, dx = x\ln x - x$. So

$$\int \ln x\, dx = x\ln x - x$$

The second subproblem is a piece-of-cake:

$$\int x\, dx = \frac{x^2}{2} \quad .$$

The third subproblem

$$\int \sin^2 x\, dx \quad ,$$

you do by **trig. identity** $\sin^2 x = \frac{1-\cos 2x}{2}$, so

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx = \frac{1}{2}(x - \frac{\sin 2x}{2}) = \frac{x}{2} - \frac{\sin 2x}{4} \quad .$$

Going back to the main problem, we have

$$\int \left(3\ln x + x + \sin^2 x\right)\, dx = 3x(\ln x - 1) + \frac{x^2}{2} + \frac{x}{2} - \frac{\sin 2x}{4} + C \quad .$$

**9**. (10 pts, 5 each) (a) Evaluate

$$\int \frac{dx}{\sqrt{x^2+2x+2}} \quad .$$

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**Ans to (a)** $\ln|x+1+\sqrt{x^2+2x+2}|+C \quad .$

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We first **complete the square**:

$$x^2+2x+2=(x+1)^2+1 \qquad .$$

Writing our integral as

$$\int \frac{dx}{\sqrt{(x+1)^2+1}} \quad .$$

We now make the straightforward substitution $u=x+1$ (so $dx=du$) getting

$$\int \frac{du}{\sqrt{u^2+1}} \quad .$$

Now we do trig. substitution $u=\tan\theta$, $du=\sec^2\theta d\theta$ and via the famous trig. identity: $u^2+1=\tan^2\theta+1=\sec^2\theta$ so $\sqrt{u^2+1}=\sec\theta$. In the $\theta$-language, our integral is

$$\int \frac{\sec^2\theta}{\sec\theta}\,d\theta=\int \sec\theta d\theta \quad .$$

From the database in your brain this equals

$$\ln|\tan\theta+\sec\theta| \quad .$$

But $\tan\theta=u$ and $\sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+u^2}$. So in the $u$-language the answer is

$$\ln|u+\sqrt{u^2+1}| \quad .$$

**Finally**, going back to the $x$-language, we get

$$\ln|x+1+\sqrt{(x+1)^2+1}|=\ln|x+1+\sqrt{x^2+2x+2}|+C \quad .$$

(b) Evaluate

$$\int \cos^6 x \sin^3 x \, dx \quad .$$

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**Ans to (b)** $\frac{1}{9}\cos^9 x - \frac{1}{7}\cos^7 x + C$

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Since the power of $\sin x$ is **odd** we use trig. substitution with $u = \cos x$. $\frac{du}{dx} = -\sin x$, $dx = -\frac{du}{\sin x}$. Our integral is

$$\int \cos^6 x \sin^3 x \, dx = \int u^6 \sin^3 x \cdot \left(-\frac{du}{\sin x}\right) = -\int u^6 \sin^2 x \, du \quad .$$

To get rid of $\sin^2 x$ we use the famous trig. identity $\sin^2 x = 1 - \cos^2 x = 1 - u^2$, getting a pure $u$-integral

$$-\int u^6(1-u^2)\, du = \int (u^8 - u^6)\, du = \frac{u^9}{9} - \frac{u^7}{7} \quad .$$

**Finally**, going back to the $x$-language, we get

$$\frac{\cos^9 x}{9} - \frac{\cos^7 x}{7} + C \quad .$$

**10**. (10 points) Evaluate

$$\int \tan^5 x\, dx \quad ,$$

using the reduction formula

$$\int \tan^m x\, dx \;=\; \frac{\tan^{m-1} x}{m-1} \;-\; \int \tan^{m-2} x\, dx \quad .$$

(**Reminder**: $\int \tan x\, dx = \ln|\sec x| + C$.)

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**Ans.** $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C \quad .$

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We plug-in $m = 5$ into the reduction formula, getting:

$$\int \tan^5 x\, dx \;=\; \frac{\tan^4 x}{4} \;-\; \int \tan^3 x\, dx \quad .$$

We now need to do the **subproblem** $\int \tan^3 x\, dx$, and for that use the reduction formula one more time, this time with $m = 3$:

$$\int \tan^3 x\, dx \;=\; \frac{\tan^2 x}{2} \;-\; \int \tan x\, dx \quad .$$

From the database in you head, or using the reminder, $\int \tan x\, dx = \ln|\sec x|$, so

$$\int \tan^3 x\, dx \;=\; \frac{\tan^2 x}{2} \;-\ln|\sec x| \quad .$$

Going back to the **main problem**, we have:

$$\int \tan^5 x\, dx \;=\; \frac{\tan^4 x}{4} \;-\left(\frac{\tan^2 x}{2} \;-\ln|\sec x|\right) \;=\; \frac{\tan^4 x}{4} \;-\; \frac{\tan^2 x}{2} \;+\ln|\sec x| + C$$

**Warning**: Many people do a very **gross** algebra mistake when they have something like

$$A - (B - C)$$

they put it as $A - B - C$. **WRONG! WATCH OUT!**, the true thing is:

$$A - B + C$$

(when you open-up parantheses, minus times minus is plus!).