Problem Type P6.1: Integrate expressions involving $\sqrt{a^2 - x^2}$, or $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, where $a$ is some number.

Example Problem P6.1: Evaluate the integral

$$\int \sqrt{1 - x^2} \, dx$$

Steps

1. If $\sqrt{a^2 - x^2}$ shows up use the substitution $x = a \sin \theta$ and you’ll have to use

$$1 - \sin^2 \theta = \cos^2 \theta .$$

If $\sqrt{a^2 + x^2}$ shows up use the substitution $x = a \tan \theta$, and you’ll have to use

$$1 + \tan^2 \theta = \sec^2 \theta .$$

If $\sqrt{x^2 - a^2}$ shows up use the substitution $x = a \sec \theta$, and you’ll have to use

$$\sec^2 \theta - 1 = \tan^2 \theta .$$

Get some trig integral of the type of Lecture 5.

Example

1. Use the substitution $x = \sin \theta$, implying $dx = \cos \theta \, d\theta$.

$$\int \sqrt{1 - x^2} \, dx = \int \sqrt{1 - \sin^2 \theta} \cos \theta \, d\theta$$

$$= \int \cos \theta \cos \theta \, d\theta = \int \cos \theta \, d\theta = \int \cos^2 \theta \, d\theta .$$

2. Use the know-how of Lecture 5 to evaluate this trig integral.

$$\int \cos^2 \theta \, d\theta = \int \frac{1 + \cos 2\theta}{2} \, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} .$$
3. Convert the answer back to the $x$ language.

3. Since $x = \sin \theta$, we have $\theta = \sin^{-1} x$, so a correct, but simplistic answer is

$$(\sin^{-1} x)/2 + (\sin 2(\sin^{-1} x))/4$$

But it is better to convert the $\sin 2\theta$ into $2 \sin \theta \cos \theta$ giving

$$(\sin 2\theta)/4 = (1/2) \sin \theta \cos \theta = (1/2)(x) \sqrt{1 - x^2}$$.

**Ans.:** $\frac{1}{2}(\sin^{-1} x) + \frac{1}{2}x \sqrt{1 - x^2}$. 