Dr. Z's Calc2 Handout for Lecture 6: Trigonometric Substitution

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Problem Type P6.1: Integrate expressions involving $\sqrt{a^2 - x^2}$, or $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, where a is some number.

Example Problem P6.1: Evaluate the integral

$$\int \sqrt{1 - x^2}\ dx$$

Steps

1. If $\sqrt{a^2 - x^2}$ shows up use the substitution $x = a \sin\theta$ and you'll have to use

$$1 - \sin^2\theta = \cos^2\theta \quad .$$

If $\sqrt{a^2 + x^2}$ shows up use the substitution $x = a \tan\theta$, and you'll have to use

$$1 + \tan^2\theta = \sec^2\theta \quad .$$

If $\sqrt{x^2 - a^2}$ shows up use the substitution $x = a \sec\theta$. and you'll have to use

$$\sec^2\theta - 1 = \tan^2\theta \quad .$$

Get some trig integral of the type of Lecture 5.

2. Use the know-how of Lecture 5 to evaluate this trig integral.

Example

1. Use the substitution $x = \sin\theta$, implying $dx = \cos\theta\ d\theta$.

$$\int \sqrt{1 - x^2}\ dx = \int \sqrt{1 - \sin^2\theta}\ \cos\theta\ d\theta$$

$$= \int \sqrt{\cos^2\theta}\cos\theta\ d\theta = \int \cos\theta\ \cos\theta\ d\theta = \int \cos^2\theta\ d\theta \quad .$$

2.

$$\int \cos^2\theta\ d\theta = \int \frac{1 + \cos 2\theta}{2}\ d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} \quad .$$

3. Convert the answer back to the x language.

3. Since $x = \sin\theta$, we have $\theta = \sin^{-1} x$, so a correct, but simplistic answer is

$$(\sin^{-1} x)/2 + (\sin 2(\sin^{-1} x))/4 \quad .$$

But it is better to convert the $\sin 2\theta$ into $2\sin\theta\cos\theta$ giving

$$(\sin 2\theta)/4 = (1/2)\sin\theta\cos\theta = (1/2)(x)\sqrt{1-x^2} \quad .$$

Ans.: $\frac{1}{2}(\sin^{-1} x) + \frac{1}{2}x\sqrt{1-x^2}$.