Problem Type P22.1:

A bacteria culture starts with $B_0$ bacteria and grows at a rate proportional to its size. After $T_1$ hours there are $B_1$ bacteria.

(a) Find an expression for the number of bacteria after $t$ hours.

(b) Find the number of bacteria after $T_2$ hours.

(c) Find the rate of growth after $T_2$ hours.

(d) When will the population reach $B_2$?

Example Problem P22.1:

A bacteria culture starts with 50 bacteria and grows at a rate proportional to its size. After 3 hours there are 800 bacteria.

(a) Find an expression for the number of bacteria after $t$ hours.

(b) Find the number of bacteria after 4 hours.

(c) Find the rate of growth after 4 hours.

(d) When will the population reach 3000?

Steps

1. Let $P(t)$ be the number of bacteria after $t$ hours. The initial value-diff. eq. is

$$P'(t) = kP(t) , \quad P(0) = B_0 ,$$

where $k$ is some yet-to-be-determined number. The solution that should be memorized is

$$P(t) = B_0e^{kt} .$$

Example

1. The initial value-diff. eq. is

$$P'(t) = kP(t) , \quad P(0) = 50 ,$$

where $k$ is some yet-to-be-determined number. The solution is

$$P(t) = 50e^{kt} .$$
2. Plug-in \( t = T_1 \) and use the fact that
\( P(T_1) = B_1 \), and solve for \( k \). Plug that
\( k \) that you have just found into
\( P(t) = B_0 e^{kt} \), and simplify (if possible or conve-
nient) using the ln rules
\( a \ln b = \ln(b^a) \)
and \( e^{\ln w} = w \).

**Shortcut:** In these problems, you can
circumvent \( e \) and ln and use
\[
P(t) = B_0 \left( \frac{B_1}{B_0} \right)^{t/T_1}
\]

2. Plug-in \( t = 3 \) and use the fact that
\( P(3) = 800 \), and solve for \( k \).

\[
P(3) = 800 \quad \text{means} \quad 50e^{3k} = 800,
\]
so \( e^{3k} = 800/50 = 16 \). Taking ln we get
\( \ln(e^{3k}) = \ln(16) \) so \( 3k = \ln 16 \) and
\( k = (1/3)(\ln 16) = \ln(16^{1/3}) \).

Plugging into \( P(t) = 50e^{kt} \). We get
\[
P(t) = 50e^{(\ln(16^{1/3}) t} = 50e^{\ln(16^{1/3}) t} = 50 \cdot 16^{t/3}.
\]

**Ans. to (a):**
\[
P(t) = 50 \cdot 16^{t/3}.
\]

Using the shortcut we get faster
\[
P(t) = B_0 \left( \frac{B_1}{B_0} \right)^{t/T_1} = 50 \left( \frac{800}{50} \right)^{t/3} = 50(16^{t/3}).
\]

3. Plug-in \( t = T_2 \) into the formula for
\( P(t) \) that you have just found.

3.
\[
P(4) = 50(16^{4/3})
\]

**Ans. to (b):** 50(16^{4/3}) appx. 2016.

4. Since \( P'(t) = kP(t) \), the rate of growth
after \( T_2 \) hours is \( kP(T_2) \), so multiply the
answer to (b) by \( k \) from step 2.

4. Rate of growth of bacteria after 4 hours,
\( P'(4), \) equals \( kP(4) = (1/3)(\ln 16) \cdot (50(16^{4/3}))
\]
appx. 1863 cell/hour.
5. Solve, for $t$, the equation $P(t) = B_2$

$$50(16^{t/3}) = 3000 \quad \text{means} \quad 16^{t/3} = 3000/50 = 60 \ .$$

Taking ln of both sides gives

$$\ln 16^{t/3} = \ln 60 \quad \text{i.e.} \quad (t/3) \ln 16 = \ln 60 \ ,$$

giving $t = 3(\ln 60)/\ln 16$ which is appx. 4.43 hours, or 4 hours and 26 minutes.

\textbf{Ans. to (d):} $3(\ln 60)/\ln 16$ which is appx. 4 hours and 26 minutes.

\textbf{Problem Type P22.2:} The half-life of some radioactive element is $T_{half}$ years. Suppose that we have a $M_0$-mg sample.

(a) Find the mass that remains after $t$ years.

(b) How much of the sample remains after $T_1$ years?

(c) After how long will only $M_1$ mg remain?

\textbf{Example Problem P22.2:} The half-life of cesium-137 is 30 years. Suppose that we have a 100-mg sample.

(a) Find the mass that remains after $t$ years.

(b) How much of the sample remains after 100 years?

(c) After how long will only 1 mg remain?

\textbf{Steps} \hspace{2cm} \textbf{Example}

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<td>1. The general formula for the mass after $t$ years, let’s call it $M(t)$ is $M(t) = M_0 \left( \frac{1}{2} \right)^{t/T_{half}}$</td>
<td>1. \textbf{Ans. to (a):} $M(t) = 100 \left( \frac{1}{2} \right)^{t/30}$</td>
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2. Plug $t = T_1$ into the formula for $M(t)$ that you have just found.

\[ M(100) = 100 \left( \frac{1}{2} \right)^{100/30} = 100 \left( \frac{1}{2} \right)^{10/3} \approx 9.92 \text{mg} \]

**Ans. to (b):** $100 \left( \frac{1}{2} \right)^{10/3} \approx 9.92 \text{mg}.$

3. Solve, for $t$, $M(t) = M_1$, using the expression for $M(t)$ that you found above.

3. We have to solve

\[ 100 \left( \frac{1}{2} \right)^{t/30} = 1 \ . \]

Dividing both sides by 100 we get

\[ \left( \frac{1}{2} \right)^{t/30} = 1/100 \ . \]

Taking reciprocals (optional)

\[ 2^{t/30} = 100 \ . \]

Taking ln of both sides

\[ (t/30)(\ln 2) = \ln 100 \ , \]

which gives $t = 30(\ln 100)/(\ln 2) = 60(\ln 10)/(\ln 2)$

appx. 199.3 years.

**Ans. to (c):** $60(\ln 10)/(\ln 2)$ appx. 199.3 years.

**Problem Type P22.3:** When a cold drink is taken from a refrigerator, its temperature is $T_0$ degrees. After $t_1$ minutes in a $T_{ambient}$-degree room, its temperature has increased to $T_1$ degrees.

(a) What is the temperature of the drink after $t_2$ minutes?

(b) When will its temperature be $T_2$ degrees?

**Example Problem P22.3:**

When a cold drink is taken from a refrigerator, its temperature is 5 degrees. After 25 minutes in a 20 degree room its temperature has increased to 10 degrees.
(a) What is the temperature of the drink after 50 minutes?

(b) When will its temperature be 15 degrees?

**Steps**

1. Set up **Newton’s Law of Cooling**

\[
\frac{dT}{dt} = k(T - T_{\text{ambient}}) \quad , \quad T(0) = T_0
\]

where \( k \) is a constant, and \( T_{\text{ambient}} \) is the ambient temperature. Taking \( y = T - T_{\text{ambient}} \) this becomes

\[
\frac{dy}{dt} = ky \quad , \quad y(0) = T_0 - T_{\text{ambient}}
\]

and remember that the solution is always

\[
y(t) = y(0)e^{kt}
\]

2. Find \( k \) by taking advantage of the fact that \( y(t_1) = T_1 - T_{\text{ambient}} \). Plug \( t = t_1 \) into \( y(t) \), set it equal to \( T_1 - T_{\text{ambient}} \) and solve for \( k \). Then plug that \( k \) into \( y(t) \).

**Example**

1. Taking \( y = T - 20 \) (note for later that \( T = y + 20 \)) we have

\[
\frac{dy}{dt} = ky \quad , \quad T(0) = 5 - 20 = -15
\]

whose solution is

\[
y(t) = -15e^{kt}
\]

2. When \( t \) equals 25, \( T \) equals 10 so \( y \) equals \( 10 - 20 = -10 \) and we have

\[
y(25) = -10
\]

so

\[
-15e^{25k} = -10 \quad \text{i.e.} \quad e^{25k} = \frac{10}{15} = \frac{2}{3}
\]

Taking the \( \ln \) of both sides we have

\[
25k = \ln\left(\frac{2}{3}\right) \quad \text{that gives} \quad k = \left(\frac{1}{25}\right)\ln\left(\frac{2}{3}\right) = -\left(\frac{1}{25}\right)\ln(3/2)
\]

Plugging that \( k \) back into \( y(t) \) we have a formula for \( y(t) \):

\[
y(t) = -15e^{-t/25}\ln(3/2)
\]
3. Plug $t = T_2$ into $y(t)$ to get $y$ at that time, and add $T_{ambient}$ to get the actual temperature.

$$y(50) = -15e^{(-50/25)\ln(3/2)} = -15e^{-\ln(3/2)} = -15e^{\ln(3/2)^{-2}} =$$

$$= -15\cdot(3/2)^{-2} = -15\cdot(2/3)^{2} = -15\cdot(4/9) = -20/3 = -\frac{2}{3}.$$ 

To get the actual temperature we add 20, getting $T(50) = 20 - 6\frac{2}{3} = 13\frac{1}{3}.$

**Ans. to (a):** The temperature of the drink after 50 minutes was $13\frac{1}{3}^\circC$.

4. Set $y(t)$ equal to $T_2 - T_{ambient}$ and solve for $t$.

$$y(t) = T_2 - T_{ambient} = 15 - 20 = -5$$

which is the same as

$$\frac{1}{3} = e^{(-t/25)\ln(3/2)}.$$ 

Taking reciprocals

$$3 = e^{(t/25)\ln(3/2)}.$$ 

Taking ln:

$$\ln 3 = (t/25)\ln(3/2).$$ 

Solving for $t$ we get

$$t = 25\cdot\ln 3/\ln(3/2) \text{ app. } 67.74\text{ mins}.$$ 

**Ans. to (b):** The temperature of the drink will be $15^\circC$ after $25(\ln 3)/\ln(3/2)$ minutes which is approximately 67.74 minutes.