Problem Type P22.1:

A bacteria culture starts with $B_0$ bacteria and grows at a rate proportional to its size. After $T_1$ hours there are $B_1$ bacteria.

(a) Find an expression for the number of bacteria after $t$ hours.

(b) Find the number of bacteria after $T_2$ hours.

(c) Find the rate of growth after $T_2$ hours.

(d) When will the population reach $B_2$?

Example Problem P22.1:

A bacteria culture starts with 50 bacteria and grows at a rate proportional to its size. After 3 hours there are 800 bacteria.

(a) Find an expression for the number of bacteria after $t$ hours.

(b) Find the number of bacteria after 4 hours.

(c) Find the rate of growth after 4 hours.

(d) When will the population reach 3000?

Steps

1. Let $P(t)$ be the number of bacteria after $t$ hours. The initial value-diff. eq. is

$$P'(t) = kP(t) \quad , \quad P(0) = B_0 \quad ,$$

where $k$ is some yet-to-be-determined number. The solution that should be memorized is

$$P(t) = B_0 e^{kt} \quad .$$

Example

1. The initial value-diff. eq. is

$$P'(t) = kP(t) \quad , \quad P(0) = 50 \quad ,$$

where $k$ is some yet-to-be-determined number. The solution is

$$P(t) = 50e^{kt} \quad .$$
2. Plug-in $t = T_1$ and use the fact that $P(T_1) = B_1$, and solve for $k$. Plug that $k$ that you have just found into $P(t) = B_0e^{kt}$, and simplify (if possible or convenient) using the ln rules $a \ln b = \ln(b^a)$ and $e^{\ln w} = w$.

**Shortcut:** In these problems, you can circumvent $e$ and $\ln$ and use

$$P(t) = B_0 \left( \frac{B_1}{B_0} \right)^{t/T_1}$$

2. Plug-in $t = 3$ and use the fact that $P(3) = 800$, and solve for $k$.

$$P(3) = 800 \quad \text{means} \quad 50e^{3k} = 800,$$

so $e^{3k} = 800/50 = 16$. Taking ln we get $\ln(e^{3k}) = \ln(16)$ so $3k = \ln 16$ and $k = (1/3)(\ln 16) = \ln(16^{1/3})$.

Plugging into $P(t) = 50e^{kt}$. We get

$$P(t) = 50e^{(\ln(16^{1/3}))t} = 50(e^{\ln(16^{1/3})})t = 50 \cdot 16^{t/3} \quad .$$

**Ans. to (a):**

$$P(t) = 50 \cdot 16^{t/3} \quad .$$

Using the shortcut we get faster

$$P(t) = B_0 \left( \frac{B_1}{B_0} \right)^{t/T_1} = 50 \left( \frac{800}{50} \right)^{t/3} = 50(16^{t/3}) \quad .$$

3. Plug-in $t = T_2$ into the formula for $P(t)$ that you have just found.

3. $P(4) = 50(16^{4/3}) \quad .$

**Ans. to (b):** $50(16^{4/3})$ appx. 2016.

4. Since $P'(t) = kP(t)$, the rate of growth after $T_2$ hours is $kP(T_2)$, so multiply the answer to (b) by $k$ from step 2.

4. Rate of growth of bacteria after 4 hours, $P'(4)$, equals $kP(4) = (1/3)(\ln 16) \cdot (50(16^{4/3}))$ appx. 1863 cell/hour.
5. Solve, for \( t \), the equation \( P(t) = B_2 \)

\[
50(16^{t/3}) = 3000 \quad \text{means} \quad 16^{t/3} = 3000/50 = 60.
\]

Taking \( \ln \) of both sides gives

\[
\ln 16^{t/3} = \ln 60 \quad \text{i.e.} \quad (t/3) \ln 16 = \ln 60,
\]

giving \( t = 3(\ln 60)/(\ln 16) \) which is appx. 4.43 hours, or 4 hours and 26 minutes.

\textbf{Ans. to (d):} \( 3(\ln 60)/(\ln 16) \) which is appx. 4 hours and 26 minutes.

\textbf{Problem Type P22.2:} The half-life of some radioactive element is \( T_{\text{half}} \) years. Suppose that we have a \( M_0 \)-mg sample.

(a) Find the mass that remains after \( t \) years.

(b) How much of the sample remains after \( T_1 \) years?

(c) After how long will only \( M_1 \) mg remain?

\textbf{Example Problem P22.2:} The half-life of cesium-137 is 30 years. Suppose that we have a 100-mg sample.

(a) Find the mass that remains after \( t \) years.

(b) How much of the sample remains after 100 years?

(c) After how long will only 1 mg remain?

\begin{center}
<table>
<thead>
<tr>
<th>Steps</th>
<th>Example</th>
</tr>
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| 1. The general formula for the mass after \( t \) years, let’s call it \( M(t) \) is | 1. Ans. to (a):
| \( M(t) = M_0 \left( \frac{1}{2} \right)^{t/T_{\text{half}}} \) | \( M(t) = 100 \left( \frac{1}{2} \right)^{t/30} \) |
| | |
\end{center}
2. Plug \( t = T_1 \) into the formula for \( M(t) \) that you have just found.

\[
M(100) = 100 \left( \frac{1}{2} \right)^{100/30} = 100 \left( \frac{1}{2} \right)^{10/3} \text{ appx. } 9.92 \text{mg}
\]

Ans. to (b): \( 100 \left( \frac{1}{2} \right)^{10/3} \text{ appx. } 9.92 \text{mg} \).

3. Solve, for \( t \), \( M(t) = M_1 \), using the expression for \( M(t) \) that you found above.

3. We have to solve

\[
100 \left( \frac{1}{2} \right)^{t/30} = 1 .
\]

Dividing both sides by 100 we get

\[
\left( \frac{1}{2} \right)^{t/30} = 1/100 .
\]

Taking reciprocals (optional)

\[
2^{t/30} = 100 .
\]

Taking \( \ln \) of both sides

\[
(t/30)(\ln 2) = \ln 100 ,
\]

which gives \( t = 30(\ln 100)/(\ln 2) = 60(\ln 10)/(\ln 2) \) appx. 199.3 years.

Ans. to (c): \( 60(\ln 10)/(\ln 2) \text{ appx. 199.3 years} \).

Problem Type P22.3: When a cold drink is taken from a refrigerator, its temperature is \( T_0 \) degrees. After \( t_1 \) minutes in a \( T_{ambient} \)-degree room, its temperature has increased to \( T_1 \) degrees.

(a) What is the temperature of the drink after \( t_2 \) minutes?

(b) When will its temperature be \( T_2 \) degrees?

Example Problem P22.3:

When a cold drink is taken from a refrigerator, its temperature is 5 degrees. After 25 minutes in a 20 degree room its temperature has increased to 10 degrees.
(a) What is the temperature of the drink after 50 minutes?

(b) When will its temperature be 15 degrees?

Steps

1. Set up **Newton’s Law of Cooling**

   \[
   \frac{dT}{dt} = k(T - T_{ambient}) \quad , \quad T(0) = T_0
   \]

   where \( k \) is a constant, and \( T_{ambient} \) is the ambient temperature. Taking \( y = T - T_{ambient} \) this becomes

   \[
   \frac{dy}{dt} = ky \quad , \quad y(0) = T_0 - T_{ambient}
   \]

   and remember that the solution is always

   \[
   y(t) = y(0)e^{kt}
   \]

2. Find \( k \) by taking advantage of the fact that \( y(t_1) = T_1 - T_{ambient} \). Plug \( t = t_1 \) into \( y(t) \), set it equal to \( T_1 - T_{ambient} \) and solve for \( k \). Then plug that \( k \) into \( y(t) \).

Example

1. Taking \( y = T - 20 \) (note for later that \( T = y + 20 \)) we have

   \[
   \frac{dy}{dt} = ky \quad , \quad y(0) = 5 - 20 = -15
   \]

   whose solution is

   \[
   y(t) = -15e^{kt}
   \]

2. When \( t \) equals 25, \( T \) equals 10 so \( y \) equals 10 - 20 = -10 and we have

   \[
   y(25) = -10
   \]

   so

   \[
   -15e^{25k} = -10 \quad \text{i.e.} \quad e^{25k} = \frac{10}{15} = \frac{2}{3}
   \]

   Taking the ln of both sides we have

   \[
   25k = \ln(\frac{2}{3}) \quad \text{that gives} \quad k = \frac{1}{25}\ln(\frac{2}{3}) = -\frac{1}{25}\ln(\frac{3}{2})
   \]

   Plugging that \( k \) back into \( y(t) \) we have a formula for \( y(t) \):

   \[
   y(t) = -15e^{(-t/25)\ln(3/2)}
   \]
3. Plug $t = T_2$ into $y(t)$ to get $y$ at that time, and add $T_{ambient}$ to get the actual temperature.

$$y(50) = -15e^{(-5/25)\ln(3/2)} = -15e^{-2\ln(3/2)} = -15e^{\ln(3/2)^{-2}} =$$
$$= -15\cdot(3/2)^{-2} = -15\cdot(2/3)^2 = -15\cdot(4/9) = -20/3 = -6\frac{2}{3}.$$  

To get the actual temperature we add 20, getting $T(50) = 20 - 6\frac{2}{3} = 13\frac{1}{3}$.  

**Ans. to (a):** The temperature of the drink after 50 minutes was $13\frac{1}{3}^\circ C$.

4. Set $y(t)$ equal to $T_2 - T_{ambient}$ and solve for $t$.

$$-5 = -15e^{(-t/25)\ln(3/2)} ,$$

which is the same as

$$\frac{1}{3} = e^{(-t/25)\ln(3/2)} .$$

Taking reciprocals

$$3 = e^{(t/25)\ln(3/2)} .$$

Taking ln:

$$\ln 3 = (t/25)\ln(3/2) .$$

Solving for $t$ we get

$$t = 25 \cdot \ln 3/ \ln(3/2) \quad \text{app. 67.74 mins}.$$  

**Ans. to (b):** The temperature of the drink will be $15^\circ C$ after $25(\ln 3)/\ln(3/2)$ minutes which is approximately 67.74 minutes.