Problem Type P21.1: Show that $y = f(x)$ is a solution of the initial value problem

$$y' + A(x)y = B(x) \quad y(a) = b$$

Example Problem P21.1: Show that $y = \sin x \cos x - \cos x$ is a solution of the initial value problem

$$y' + (\tan x)y = \cos^2 x \quad y(0) = -1$$

Steps

1. Copy $y$, and compute $y'$ (and $y''$ etc., if necessary)

   1. $y = \sin x \cos x - \cos x$

      $$y' = (\sin x \cos x - \cos x)'$$

      $$= (\sin x)'(\cos x) + (\sin x)(\cos x)' - (\cos x)'$$

      $$= (\cos x)(\cos x) + (\sin x)(\sin x) + \sin x = \cos^2 x - \sin^2 x + \sin x$$

2. Substitute $y$ and $y'$ (and if applicable, $y''$ etc.) into the diff. eq. and see if the right side equals the left side.

   2. $y' + (\tan x)y = \cos^2 x - \sin^2 x + \sin x + (\tan x)(\sin x \cos x - \cos x)$

      $$\cos^2 x - \sin^2 x + \sin x + \sin x (\sin x - 1) \cos x =$$

      $$\cos^2 x - \sin^2 x + \sin x + \sin x (\sin x - 1) =$$

      $$\cos^2 x - \sin^2 x + \sin x + \sin^2 x - \sin x$$

      $$= \cos^2 x$$

      which is as claimed.

3. Plug the initial condition $x = a$ into the function, and check that it is equal to $b$.

   3. When $x = 0$,

      $$y = (\sin 0)(\cos 0) - (\cos 0) = 0 - 1$$

      $$= -1$$

      as claimed.
**Problem Type P21.2:** For what value of $r$ does the function $y = e^{rt}$ satisfy the differential equation $y'' + ay' + by = 0$?

**Example Problem P21.2:** For what value of $r$ does the function $y = e^{rt}$ satisfy the differential equation $y'' + y' - 2y = 0$?

**Steps**

1. Pretend that $r$ is a specific number, copy $y$, and compute $y'$ and $y''$. These expressions involve $r$ as well as $t$.

2. Substitute $y, y'$ and $y''$ into the diff. eq. Then factor out $e^{rt}$.

3. Set this equal to 0 and solve for $r$. Plug these values of $r$ into $y = e^{rt}$.

**Example**

1. $y = e^{rt}$, $y' = re^{rt}$, $y'' = r^2 e^{rt}$.

2. $y'' + y' - 2y = r^2 e^{rt} + re^{rt} - 2e^{rt} = (r^2 + r - 2)e^{rt}$.

3. Since $e^{rt}$ is never zero, $(r^2 + r - 2)e^{rt} = 0$ means $r^2 + r - 2 = 0$, and factoring $(r + 2)(r - 1) = 0$ yields the roots $r = -2$ and $r = 1$. So the solutions of the given diff. eq. of the suggested form are $y = e^{-2t}$ and $y = e^{t}$.

**Ans.:** $y = e^{-2t}$ and $y = e^{t}$.

**Problem Type P21.3:** Solve the differential equation

$$y' = \frac{A(x)}{B(y)} \quad \text{or} \quad y' = A(x)B(y) \quad \text{etc.}$$

**Example Problem P21.3:** Solve the differential equation

$$y' = y^2 \sec x$$
Steps

1. If not already written like this, replace \( y' \) by \( \frac{dy}{dx} \). Treat \( dy \) and \( dx \) as algebraic quantities and separate the \( x \) part from the \( y \) part.

\[
\frac{dy}{dx} = \frac{A(x)}{B(y)} \quad \text{or} \quad \frac{dy}{dx} = A(x)B(y) \quad \text{etc.}
\]

implies

\[
B(y)dy = A(x)dx \quad \text{respectively}
\]

\[
\frac{dy}{B(y)} = A(x)dx \quad \text{etc.}
\]

2. Apply the Integral sign to both sides, and perform the integration. Only put the +C on one side.

\[
\int y^{-2} \, dy = \int \sec x \, dx ,
\]

gives

\[
-\frac{1}{y} = \ln |\sec x + \tan x| + C
\]

3. If possible, solve for \( y \). Otherwise leave it in implicit form. If there is an initial condition then plug it in and solve for \( C \). If nothing is mentioned (like in this problem), then leave \( C \) alone.

\[
y = \frac{-1}{\ln |\sec x + \tan x| + C}.
\]

Ans.: \( y = \frac{-1}{\ln |\sec x + \tan x| + C} \).

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**Problem Type P21.4:** Find an equation of the curve that passes through the point \((a, b)\) and whose slope at \((x, y)\) is \(A(y)/B(x)\).

**Example Problem P21.4:** Find an equation of the curve that passes through the point \((1, 1)\) and whose slope at \((x, y)\) is \(y^2/x^3\).
1. Slope is derivative, so set it equal to \( \frac{dy}{dx} \). Treat \( dy \) and \( dx \) as algebraic quantities and separate the \( x \) part from the \( y \) part.

\[
\frac{dy}{dx} = \frac{A(y)}{B(x)}
\]

implies

\[
\frac{dy}{A(y)} = \frac{dx}{B(x)}
\]

2. Apply the Integral sign to both sides, and perform the integration. Only put the \( +C \) on one side.

\[
\int \frac{dy}{y^2} = \int \frac{dx}{x^3}
\]

which is the same as

\[
y^{-2} \, dy = x^{-3} \, dx
\]

3. Plug in the point \((x = a, y = b)\) and solve for \( C \). Plug back that value for \( C \) and try to express \( y \) in terms of \( x \) if possible. Otherwise leave it in implicit form.

\[
y^{-1} = \frac{x^{-2}}{-2} + C
\]

which gives

\[
\frac{-1}{y} = \frac{-1}{2x^2} + C
\]

3. 
\[
\frac{-1}{1} = \frac{-1}{2 \cdot 1^2} + C
\]

giving \( C = -1/2 \). Incorporating that \( C \) gives

\[
\frac{-1}{y} = \frac{-1}{2x^2} - \frac{1}{2}
\]

and algebra gives

\[
y = \frac{2x^2}{1 + x^2}
\]

Ans.: \( \frac{2x^2}{1 + x^2} \).