Dr. Z's Calc2 Handout For Lecture 2 [Volumes and Average of a Function]

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Problem Type P2.1: Find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis

$$y = f(x), x = a, x = b, y = 0$$

Example Problem P2.1: Find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis

$$y = 1/x^2, x = 1, x = 2, y = 0$$

Steps

1. Sketch the region and see whether the bottom is the $x - axis$, if it is, then the volume is

$$\pi \int_a^b f(x)^2 dx$$

Example

1. Indeed, since $y = 0$ (alias the x-axis) is one of the bounding lines, the volume is simply

$$\pi \int_1^2 (\frac{1}{x^2})^2 dx = \pi \int_1^2 x^{-4} dx = \pi \frac{x^{-3}}{-3}\Big|_1^2 = -\frac{\pi}{3}\frac{1}{x^3}\Big|_1^2 =$$

$$-\frac{\pi}{3}(\frac{1}{2^3} - \frac{1}{1^3}) = \frac{7\pi}{24}$$

Ans.: $\frac{7\pi}{24}$.

Problem Type P2.2: Find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis

$$y = f(x), x = g(y)$$

Example Problem P2.2: Find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis

$$y = x^2, y^2 = x$$

Steps

1. Sketch the two curves and see where they intersect by solving for x and y. Then decide who is TOP and who is BOTTOM, and write both curves as $y = Expression(x)$. The x-coordinates of these points of intersection are your *limit of integration.* (if there are more than two points of intersection you have to split the interval to two or more intervals like in Problem P1.2 for areas)

2. If the limits of integration are $x = a$ and $x = b$, then the volume is

$$\pi \int_a^b (TOP^2 - BOTTOM^2)dx$$

3. Evaluate the integral

Example

1. Solving $\{y^2 = x, y = x^2\}$ yields $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$ as points of intersection, so the limits of integration are $x = 0$ and $x = 1$. The TOP is the curve $x = y^2$ and the relevant portion, in the x-language, is $y = \sqrt{x}$. The BOTTOM is $y = x^2$.

2. Since, here $a = 0$, $b = 1$ $TOP = \sqrt{x}$, $BOTTOM = x^2$, the volume is

$$\pi \int_0^1 ((\sqrt{x})^2 - (x^2)^2)dx$$

3.

$$Volume = \pi \int_0^1 (x - x^4)dx = \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right) \Big|_0^1$$

$$= \pi \left((\frac{1^2}{2} - \frac{1^5}{5}) - 0 \right) =$$

$$\frac{3\pi}{10} \quad .$$

Ans.: $\frac{3\pi}{10}$.

Problem Type P2.3: Find the volume of the solid obtained by rotating the region bounded by $y = f(x)$, $y = g(x)$ about the axis $y = A$.

Example P2.3: Find the volume of the solid obtained by rotating the region bounded by $y = x^4$,

$y = 1$ about $y = 2$.

Steps

1. Convert everything to the x-language (if necessary) getting two functions of x (in some cases one of them may be a constant, like in this problem). Set them equal to each other to see where they meet. Sketch the region and see who is on top and who is at the bottom, and find the limits of integration, by setting the two functions of x equal to each other.

2. If the limits of integration are $x = a$ and $x = b$, then the volume is

$$\pi \int_a^b ((FAR - A)^2 - (CLOSER - A)^2)dx$$

3.

Evaluate the integral

Example

1. Setting x^4 and 1 equal to each other yields $x^4 = 1$, which is the same as $x^4 - 1 = 0$, and factoring $(x^2 - 1)(x^2 + 1) = (x-1)(x+1)(x^2+1) = 0$ giving the two solutions $x = -1$ and $x = 1$. These are your *limits of integration.* From the sketch (do it!) $y = x^4$ is FARTHER away from the *given axis of rotation* ($y = 2$ in this problem) while $y = 1$ is CLOSER to it.

2.

Since, here $a = -1$, $b = 1$ $FAR = x^4$, $CLOSER = 1$, the volume is

$$Volume = \pi \int_{-1}^{1} ((x^4 - 2)^2 - (1 - 2)^2)dx$$

3.

$$Volume = \pi \int_{-1}^{1} ((x^4-2)^2-(1-2)^2)dx = \pi \int_{-1}^{1} (x^8-4x^4+3)dx =$$

$$\pi(\frac{x^9}{9} - 4\frac{x^5}{5} + 3x)\Big|_{-1}^{1} = \frac{208\pi}{45} \quad .$$

Ans.: $\frac{208\pi}{45}$.

Tip: If you get a negative answer, it (probably) means that you reversed FARTHER and CLOSER, and the answer is the absolute value. Remember: Volume can never be negative!

Note: If there are more than two points of intersection you have to break-up the problem to two (or more) intervals like in Problem P1.2.

Note: If the axis of rotation is of the form $x = A$ (i.e. parallel to the y-axis, in particular the

y-axis itself) everything is analogous, but you express all curves in the y-language, i.e. your curves are of the form $x = Expression(y)$ and the integration is w.r.t. to dy.

Average Value of a Function

Problem Type P2.4: Find the average value of a function $f(x)$ in the interval $[a, b]$ and find c such that $f(c) = f_{ave}$.

Example Problem P2.4: Find the average value of $f(x) = (x-2)^2$ in the interval $[1, 4]$ and find c such that $f(c) = f_{ave}$.

Steps

1. Use the formula

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

2. Evaluate the integral

3. Solve, for c, $f(c) = f_{ave}$. Only retain the solutions that lie in the interval $[a, b]$.

Example

1. Set up the integral

$$f_{ave} = \frac{1}{4-1}\int_1^4 (x-2)^2 dx$$

2.

$$f_{ave} = \frac{1}{4-1}\int_1^4 (x-2)^2 dx = \frac{1}{3}\int_1^4 (x-2)^2 dx$$

$$= \frac{1}{3}\left(\frac{(x-2)^3}{3}\Big|_1^4\right) \frac{1}{3}\left(\frac{(4-2)^3}{3} - \frac{(1-2)^3}{3}\right) = 1.$$

3. We have to solve $(c-2)^2 = 1$, i.e. $c - 2 = \pm 1$ giving the solutions $c = 3$ and $c = 1$. In this case they both lie there, so **Ans.**: $f_{ave} = 1$; $c = 1$ and $c = 3$.