Dr. Z's Calc2 Handout for Lecture 16 [Taylor Series]

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Problem Type P16.1: Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series.

Example Problem P16.1: Find the Maclaurin series for $f(x) = \sin x$ using the definition of a Maclaurin series.

Steps	**Example**
1. Find the first few derivatives of $f(x)$. Then plug-in, $x = 0$.	**1.** $f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x,$ $f'''(x) = -\cos x, f^{(4)}(x) = \sin x, f^{(5)}(x) = \cos x \ldots$ Plugging-in $x = 0$ we get $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1,$ $f^{(4)}(0) = 0, f^{(5)}(0) = 1, \ldots$
2. Write-down the general formula for the Maclaurin series and plug-in the values above. $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad .$$	**2.** $$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \frac{f^{(5)}(0)}{5!} x^5 + \ldots$$ $$= \frac{0}{0!} + \frac{1}{1!} x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5 + \ldots$$ $$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

3. If possible, detect a pattern and write the general series.

3.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Problem Type P16.2: Find the Taylor series for $f(x)$ centered at the given value of a.

Example Problem P16.2: Find the Taylor series for $f(x) = \sin x$ centered at $a = \pi/2$.

Steps | **Example**

1. Find the first few derivatives of $f(x)$. Then plug-in, $x = a$.

1.

$$f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x,$$

$$f'''(x) = -\cos x, f^{(4)}(x) = \sin x, f^{(5)}(x) = \cos x \ldots$$

Plugging-in $x = \pi/2$ we get

$$f(\pi/2) = 1, f'(\pi/2) = 0, f''(\pi/2) = -1, f'''(\pi/2) = 0,$$

$$f^{(4)}(\pi/2) = 1, f^{(5)}(\pi/2) = 0, \ldots$$

2. Write-down the general formula for the Taylor series centered at $x = a$ and plug-in the values above.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad .$$

2.

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)}{n!}(x-\pi/2)^n =$$

$$\frac{f^{(0)}(\pi/2)}{0!} + \frac{f^{(1)}(\pi/2)}{1!}(x-\pi/2) + \frac{f^{(2)}(\pi/2)}{2!}(x-\pi/2)^2 +$$

$$\frac{f^{(3)}(\pi/2)}{3!}(x-\pi/2)^3+\frac{f^{(4)}(\pi/2)}{4!}(x-\pi/2)^4+$$

$$\frac{f^{(5)}(\pi/2)}{5!}(x-\pi/2)^5+\dots$$

$$=\frac{1}{0!}+\frac{0}{1!}(x-\pi/2)+\frac{-1}{2!}(x-\pi/2)^2+\frac{0}{3!}(x-\pi/2)^3+$$

$$\frac{1}{4!}(x-\pi/2)^4+\frac{0}{5!}(x-\pi/2)^5+\dots$$

$$=1-\frac{(x-\pi/2)^2}{2!}+\frac{(x-\pi/2)^4}{4!}-\frac{(x-\pi/2)^6}{6!}+\dots$$

3. If possible, detect a pattern and write the general series.

3.

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{(x-\pi/2)^{2n}}{(2n)!}$$

Problem Type P16.3: Use known Maclaurin series to obtain the Maclaurin series for $f(x)$, where $f(x)$ is a product and/or composition of standard functions.

Example Problem P16.3: Use known Maclaurin series to obtain the Maclaurin series for $f(x) = x^3e^{-4x}$.

Steps

1. Decide who is (or are) the most important function in the expression, and write down its (their) Maclaurin series, using w rather than x.

Example

1. $f(x) = x^3e^{-4x}$ features the exponential function. Recall that

$$e^w=\sum_{n=0}^{\infty}\frac{w^n}{n!} \quad .$$

2. Find out what's inside the important function and plug-in for w the needed quantity.

2. Plugging-in $w = -4x$ into the exponential series, we get

$$e^{-4x} = \sum_{n=0}^{\infty} \frac{(-4x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} x^n$$

3. Use series manipulation to finish it up.

3.

$$f(x) = x^3 e^{-4x} = x^3 \sum_{n=0}^{\infty} \frac{(-4x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} x^{n+3} \quad .$$

Problem Type P16.4: Use multiplication or division of power series to find the first four (or whatever) non-zero terms of the Maclaurin series for $f(x)$, where $f(x)$ is a product and/or quotinet of several standard functions.

Example Problem P16.4: Find the first four non-zero terms of the Maclaurin expansion of $e^{2x}\cos(3x)$

Steps

1. Write the first few terms of the Maclaurin series of the 'ingredients' using the formula sheet or your memory.

Example

1.

$$e^x = 1+x+x^2/2+x^3/6+\ldots \qquad \cos x = 1-x^2/2+x^4/24+\ldots$$

which yields

$$e^{2x} = 1+2x+(2x)^2/2+(2x)^3/6+\ldots = 1+2x+2x^2+(4/3)x^3+\ldots \quad ,$$

$$\cos 3x = 1-(3x)^2/2+(3x)^4/24+\ldots = 1-(9/2)x^2+(27/8)x^4+\ldots$$

2. Use algebra to multiply (or divide) the ingredients together, discarding all terms of higher order.

2.

$$e^{2x}\cos 3x$$

$= (1+2x+2x^2+(4/3)x^3+\ldots)(1-(9/2)x^2+(27/8)x^4+\ldots)$

$= (1-(9/2)x^2+(27/8)x^4+\ldots)+(2x)(1-(9/2)x^2+(27/8)x^4+\ldots)+$

$(2x^2)(1-(9/2)x^2+(27/8)x^4+\ldots)+(4/3)x^3(1-(9/2)x^2+(27/8)x^4+\ldots)+$

$= 1-(9/2)x^2+(27/8)x^4+\ldots+2x-9x^3+(27/4)x^3+\ldots$

$+2x^2-9x^4+\ldots+(4/3)x^3+\ldots$

3. Collect terms up to the desired power.

3.

$f(x) = 1+2x-(5/2)x^2+(97/12)x^3+\ldots$

Ans.: $f(x) = 1+2x-(5/2)x^2+(97/12)x^3+\ldots$.

Problem Type P16.5: Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{x^M}{a+bx^N} \quad ,$$

for integers N and M and numbers a and b.

Example Problem P16.5: Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{x^3}{4+36x^2} \quad .$$

Steps	**Example**

1. You'd like to use the famous geometrical series power series, and let's use the letter z:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

whose radius of convergence is 1, i.e. it is valid for $|z| < 1$.

With this in mind we rewrite our function of x, $f(x)$, as

$$x^M \cdot \frac{1}{a + bx^N} = x^M \cdot \frac{1}{a(1 + (b/a)x^N)} = \frac{x^M}{a} \cdot \frac{1}{1 - (-b/a)x^N} \quad .$$

1.

$$\frac{x^3}{4+36x^2} = x^3 \cdot \frac{1}{4+36x^2} = x^3 \cdot \frac{1}{4(1+9x^2)}$$

$$= \frac{x^3}{4} \cdot \frac{1}{1-(-9x^2)} \quad .$$

2. Plug-in $z = (-b/a)x^N$ into the formula

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (valid\ for\ |z| < 1)$$

to get

$$\frac{1}{1-(-b/a)x^N} = \sum_{n=0}^{\infty} ((-b/a)x^N)^n$$

$$(valid\ for\ |(-b/a)x^N| < 1)$$

Simplify!

2. Plugging $z = -9x^2$ into

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

(valid for $z| < 1$) gives

$$\frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty} (-9x^2)^n \quad ,$$

valid for $|9x^2| < 1$, which simplifies to

$$\frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty} (-9)^n x^{2n}$$

(valid for $|x^2| < 1/9$) and hence

$$\frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty} (-9)^n x^{2n}$$

(valid for $|x| < 1/3$).

3. Finish it up by multiplying both sides by $\frac{x^M}{a}$, and simplifying. Also simplify the condition of validity to get the interval of convergence.

3.

$$\frac{x^3}{4+36x^2} = \frac{x^3}{4} \cdot \frac{1}{1-(-9x^2)} =$$

$$\frac{x^3}{4}\sum_{n=0}^{\infty}(-9)^n x^{2n} = \sum_{n=0}^{\infty}\frac{(-9)^n}{4}x^{2n+3} =$$

$(1/4)x^3+(-9/4)x^5+(81/4)x^7+\ldots\,(valid\ for\ |x|<1/3)$

Now $|x| < 1/3$ is the same as the interval $(-1/3, 1/3)$.

Ans.: The power-series representation is

$$\frac{x^3}{4+36x^2} = \sum_{n=0}^{\infty}\frac{(-9)^n}{4}x^{2n+3} \quad ,$$

and the interval of convergence is $(-1/3, 1/3)$.
(the radius of convergence is $1/3$).

Problem Type P16.6: Evaluate the indefinite integral as a power series. What is the radius of convergence?

$$\int f(x)\ dx$$

where $f(x)$ is a function whose power-series representation you can find out (either from the formula sheet or by manipulating geometric series like in 11.9a).

Example Problem P16.6: Evaluate the indefinite integral as a power series. What is the radius of convergence?

$$\int \frac{x^3}{4+36x^2}\ dx$$

Steps	**Example**
1. Express the integrand as a power-series. In other words, do 11.9a.	**1.** Doing 11.9a we have $$\frac{x^3}{4+36x^2} = \sum_{n=0}^{\infty}\frac{(-9)^n}{4}x^{2n+3} \quad ,$$

2. Integrate term-by-term, using the famous formula

$$\int x^m \, dx = \frac{x^{m+1}}{m+1}$$

Do not worry about the $+C$ until the very end.

2.

$$\int \frac{x^3}{4+36x^2} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-9)^n}{4} \int x^{2n+3} \, dx \quad ,$$

$$= \sum_{n=0}^{\infty} \frac{(-9)^n}{4} \frac{x^{2n+4}}{2n+4}$$

$$= \sum_{n=0}^{\infty} \frac{(-9)^n}{4(2n+4)} x^{2n+4} \quad .$$

3. Add $+C$ at the **beginning**, and note that the radius of convergence is **always** the same as that of the integrand. We found out in 11.9a that it was 1/3, so:

3. Ans.:

$$C + \sum_{n=0}^{\infty} \frac{(-9)^n}{4(2n+4)} x^{2n+4} \quad ,$$

and the radius of convergence is 1/3.

Problem Type P16.7 (a) Expand $\sqrt[m]{a+bx^n}$ (or $1/\sqrt[m]{a+bx^n}$) as a power series.

(b) Use part (a) to estimate some function-value correct to so-and-so many deciml places.

Example Problem P16.7: (a) Expand $\sqrt[5]{1+x}$ as a power series.

(b) Use part (a) to estimate $\sqrt[5]{1.01}$ correct to six decimal places.

Steps	**Example**
1. First rewrite the function in pure exponent-notation $A(1+Bx^n)^k$ for some numbers A, B and k.	**1.** $f(x) = (1+x)^{1/5}$.

2. Write down the **Binomial Series**, either from your memory or from the formula sheet, using the variable w. Then replace w by whatever is needed to make it coincide with the $f(x)$. Spell out the first few terms.

$$(1+w)^k = \sum_{n=0}^{\infty} \binom{k}{n} w^n \quad ,$$

where

$$\binom{k}{n} = \frac{k(k-1)\dots(k-n+1)}{n!} \quad .$$

2.

$$\sqrt[5]{1+x} = (1+x)^{1/5} = \sum_{n=0}^{\infty} \binom{1/5}{n} x^n =$$

$$1+(1/5)x+\frac{(1/5)(-4/5)}{2!}x^2+\frac{(1/5)(-4/5)(-9/5)}{3!}x^3+$$

$$\frac{(1/5)(-4/5)(-9/5)(-14/5)}{4!}x^4+\dots$$

$$= 1+\frac{x}{5}x-\frac{2x^2}{25}+\frac{6x^3}{125}-\frac{21x^4}{625}+\dots$$

(this is the **Ans. to (a)**)

3. Decide which is the appropriate x to plug-in, and plug-it into the Maclaurin expansion, quit when the next-term-to-be-added (or subtracted) is less that the desired error.

3. Here $x = .01$, and we have the following approximations

$$\sqrt[5]{1.01} = 1+\frac{.01}{5}-\frac{2(.01)^2}{25}+\frac{6(.01)^3}{125}-\frac{21(.01)^4}{625}+\dots$$

$$1+(.2)10^{-2}-(.8)10^{-5}+(.48)10^{-7}+(.336)10^{-9}+\dots \quad .$$

Since our desired accuracy is 6 decimal places, it means that the allowed error is $(.5)10^{-6}$. The fourth term is already less than that, so we ignore it and anything after that, and we get

$$\sqrt[5]{1.01} \approx 1+(.2)10^{-2}-(.8)10^{-5} = 1.200200 \quad .$$

Ans. to (b): $\sqrt[5]{1.01} \approx 1.200200$

Problem Type P16.8 (a) Use the binomial series to expand functions involving square-root, like $1/\sqrt{1-x^2}$, $1/\sqrt{1+x^2}$ etc.

(b) Use part (a) to find the Maclaurin series for some inverse-trig function that is known to be the indefinite integral of the function of part (a).

Example Problem P16.8 (a) Use the binomial series to expand $1/\sqrt{1-x^2}$.

(b) Use part (a) to find the Maclaurin series for $\sin^{-1} x$.

Steps	**Example**
1. First rewrite the function in exponent notation $A(1+Bx^n)^k$ for some numbers A, B and k.	**1.** $f(x) = (1-x^2)^{-1/2}$.
2. Write down the **Binomial Series**, either from your memory or from the formula sheet, using w, and then replace w by the right monomial in x. $$(1+w)^k = \sum_{n=0}^{\infty} \binom{k}{n} w^n \quad ,$$ where $$\binom{k}{n} = \frac{k(k-1)\dots(k-n+1)}{n!} \quad .$$ Use algebra to simplify $\binom{k}{n}$.	**2.** Replacing w by $-x^2$, and k by $-1/2$ $$(1-x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \binom{-1/2}{n} (-x^2)^n$$ We have $$\binom{-1/2}{n} = \frac{(-1/2)(-3/2)\dots(-1/2-n+1)}{n!} =$$ $$\frac{(-1/2)(-3/2)\cdots(-(2n-1)/2)}{n!} =$$ $$\frac{(-1)^n(1)(3)\cdots(2n-1)}{2^n n!} =$$ And going back to the expansion of $(1-x^2)^{-1/2}$, $$(1-x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(1)(3)\cdots(2n-1)}{2^n n!} x^{2n}$$ (this is the **Ans. to (a)**)

3. Integrate term-by-term, and plug-in $x = 0$ to get C, and plug that C back.

3.

$$\sin^{-1} x = \int (1 - x^2)^{-1/2}\ dx =$$

$$C + x + \sum_{n=1}^{\infty} \frac{(1)(3)\cdots(2n-1)}{2^n n!} \int x^{2n}\ dx$$

$$C + x + \sum_{n=1}^{\infty} \frac{(1)(3)\cdots(2n-1)}{2^n n!} \frac{x^{2n+1}}{2n+1}$$

$$= C + x + \sum_{n=1}^{\infty} \frac{(1)(3)\cdots(2n-1)}{2^n n!(2n+1)} x^{2n+1}$$

When $x = 0$, $\sin^{-1}(0) = 0$ (since $\sin 0 = 0$), so $C = 0$, and we have **Ans. to (b)**:

$$= x + \sum_{n=1}^{\infty} \frac{(1)(3)\cdots(2n-1)}{2^n n!(2n+1)} x^{2n+1}$$