Dr. Z's Calc2 Handout for Lecture 14 [The Ratio and Root Tests]

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Problem Type P14.1: Determine whether the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, conditionally convergent, or divergent.

Example Problem P14.1: Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

$$(a) \quad \sum_{n=1}^{\infty} \frac{n^2}{(-2)^n} \quad ,$$

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \quad ,$$

$$(c) \sum_{n=1}^{\infty} \frac{(-2)^n n!}{n^n} \quad .$$

Steps	**Example**
1. Test for **absolute convergence** by testing whether the series $\sum_{n=1}^{\infty} \lvert a_n \rvert$ is convergent. Use any test, but the most efficient and popular ones are the **ratio test** that says that if $\lim_{n\to\infty} \lvert a_{n+1}/a_n \rvert$ is < 1 then it is abs. convergent, whereas if it is > 1 then it is divergent, and if it equals 1 exactly, then there is nothing you can say. The ratio test is analogous but with $\lim_{n\to\infty} \lvert a_n \rvert^{1/n}$ instead. If you have $n!$, or polynomials, then it is easier to use the ratio test. If you have terms like c^n, where c is a constant, then you can use either tests, and if you have terms like $(3n+5)^n/(5n+7)^n$ (but no $n!$), then it is much easier to use the root test.	**1.** For (a) we use $a_n = \frac{n^2}{(-2)^n}$, so $$\lvert a_{n+1}/a_n \rvert = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{(n+1)^2}{2n^2}$$ whose limit is $1/2$. Since this is < 1, we are done! **Ans. to (a)**: absolutely convergent. For (b) we use either test. $a_n = (-1)^{n+1} n^{-1/2}$, so $\lvert a_{n+1}/a_n \rvert = (n/(n+1))^{1/2}$, whose limit it 1. Alternatively, $\lvert a_n \rvert^{1/n} = 1/\sqrt{n^{1/n}}$, whose limit is also 1, so either way the test is inconclusive. On the other hand $\sum_{n=1}^{\infty} \lvert a_n \rvert$ is a p-series with $p = 1/2$, so we know by the p-test that it is divergent. Hence the series $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-1/2}$ is **definitely** not abs. conv. Stand by for the next step when we decide whether it is cond. conv. or outright divergenet.

For (c) $a_n = \frac{(-2)^n n!}{n^n}$, and the $n!$ calls for the ratio test.

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}}$$

$$= \frac{2(n+1)n^n}{(n+1)^{n+1}} = \frac{2n^n}{(n+1)^n} = 2\left(\frac{n}{n+1}\right)^n$$

This brings to mind the famous limit

$$\lim_{n\to\infty}((n+1)/n)^n = e$$

(it is in the formual sheet). So, for this problem,

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} 2/\left(\frac{n+1}{n}\right)^n = 2/e.$$

Since $e = 2.718\ldots$, $2/e$ is < 1 and it follows by the ratio test that we are done, and:

Ans. to (c): The series is abs. conv.

2. If we made it to this step, it means that the series is not abs. conv. Use the ping-pong test, the p-test, the integral test, or whatever to decide whether the series is convergent or divergent.

2. Only (b) survived to this step, since we know that the series is not abs. conv. But it is an alternating series, that is convergent by the ping-pong test. A series that is not abs. conv. but nevertheless conv. is called **conditionally convergent**.

Ans. to (b): The series is cond. conv.