Dr. Z's Calc2 Handout for Lecture 1 [Areas Between Curves]

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Problem Type P1.1 : Find the area of the region enclosed between $y = Expr_1(x)$ and $y = Expr_2(x)$, where they meet at two points.

Example Problem P1.1:

Find the area of the region enclosed between $y = 12 - x^2$ and $y = x^2 - 6$.

Steps	**Example**
1. Find the points of intersection by solving, for x the algebraic equation $$Expr_1(x) = Expr_2(x) \quad .$$ If you get just one root, then the region is infinite, and there is no (finite) answer. If you get two roots, let's call them $x_1 < x_2$, then the limits of the integral will be x_1 and x_2. If you get three roots, see Problem P1.2 below.	**1.** $12 - x^2 = x^2 - 6$ means $2x^2 = 18$, i.e. $2(x^2 - 9) = 0$ so $x^2 - 9 = 0$, which means $(x-3)(x+3) = 0$, and we have two roots $x = -3$ and $x = 3$.
2. For each of the intervals of integration (in this example there is just one), determine which curve in on top, by plugging-in a random value (or sketching).	**2.** Plugging-in $x = 0$ into $y = 12 - x^2$ gives $y = 12$, while plugging-in $x = 0$ into $y = x^2 - 6$ gives $y = -6$. Since 12 is bigger than -6, we have that $TOP = 12 - x^2$ and $BOT = x^2 - 6$.
3. The area is the integral $$\int_{x_1}^{x_2} [TOP(x) - BOT(x)] \quad .$$	**3.** The area is $$\int_{-3}^{3} [(12-x^2)-(x^2-6))] \quad = \int_{-3}^{3} [18-2x^2] \quad =$$ $$18x - \frac{2x^3}{3}\Big\vert_{x=-3}^{3} = 18(3-(-3)) - \frac{2(3^3-(-3)^3)}{3} = 108-36 = 72 \quad .$$ **Ans.**: 72.

Problem Type P1.2 : Find the area of the region enclosed between $y = Expr_1(x)$ and $y = Expr_2(x)$, where they meet at three points.

Example Problem P1.2:

Find the area of the region enclosed between $y = x^3 - x$ and $y = 3x$.

Steps

1. Find the points of intersection by solving, for x the algebraic equation

$$Expr_1(x) = Expr_2(x) \quad .$$

If you get one root, then the region is infinite, and there is no (finite) answer. If you get two roots, let's call them $x_1 < x_2$, then the limits of the integral will be x_1 and x_2 (like in P1.1). If you get three roots, $x_1 < x_2 < x_3$, then you need to do TWO integrals, one $\int_{x_1}^{x_2}$ and the other $\int_{x_2}^{x_3}$.

2. For each of the intervals of integration (in this example there are two: (x_1, x_2) and (x_2, x_3)), determine which curve is on top, by plugging-in a random value (or sketching).

Example

1. $x^3 - x = 3x$ means $x^3 - 4x = 0$, i.e. $x(x-2)(x+2) = 0$ so we have *three* roots $x = -2$, $x = 0$ and $x = 2$.

2. Plugging-in $x = -1$ into $y = x^3 - x$ gives $y = 0$, while plugging-in $x = -1$ into $y = 3x$ gives $y = -3$. Since 0 is bigger than -3, we have that on the interval $(-2, 0)$, $TOP = x^3 - x$ and $BOT = 3x$.

Regarding the interval $(0, 2)$, plugging-in $x = 1$ into $y = x^3 - x$ gives $y = 0$, while plugging-in $x = 1$ into $y = 3x$ gives $y = 3$. Since 3 is bigger than 0, we have that on the interval $(0, 2)$, $TOP = 3x$ and $BOT = x^3 - x$.

3. The area is the sum of the integrals

$$\int_{x_1}^{x_2} [TOP(x) - BOT(x)] +$$

$$\int_{x_2}^{x_3} [TOP(x) - BOT(x)] \quad ,$$

note that who is TOP and who is BOT (usually) gets switched between these intervals.

3. The area is

$$\int_{-2}^{0} [(x^3 - x) - (3x)] \quad + \int_{0}^{2} [(3x) - (x^3 - x)] \quad =$$

$$\int_{-2}^{0} [x^3 - 4x] \quad + \int_{0}^{2} [-(x^3 - 4x)] \quad =$$

$$[x^4/4 - 2x^2]|_{-2}^{0} + [-(x^4/4 - 2x^2)]|_{0}^{2} = 4 + 4 = 8 \quad .$$

Ans.: 8.