Solution to a typical cross section problem

1. The base of a solid is the region inside the ellipse \( x^2 + 4y^2 = 4 \). Each cross-section of the solid perpendicular to the \( y \)-axis is a square. What is the volume of the solid?

**Sol. of 1:** The ellipse crosses the \( y \)-axis at the points \((0, 1)\) and \((0, -1)\) (set \( x = 0 \) in the equation \( x^2 + 4y^2 = 4 \) and get \( 4y^2 = 4 \), so \( y = -1, 1 \)).

A cross-section perpendicular to the \( y \)-axis has length \( 2x \) (since the distance from the \( y \)-axis of a typical point \((x, y)\) is by definition of coordinates, \( x \), and it goes both ways, so by symmetry it is twice \( x \)). So the area of the square above it is \((\text{side})^2 = (2x)^2 = 4x^2\).

So the desired volume is
\[
\int_{-1}^{1} 4x^2 \, dy = 4 \int_{-1}^{1} x^2 \, dy
\]

But this, right now, does not make sense. We need an integral in terms of \( y \)!

We now use the equation of the given ellipse:

\[
x^2 + 4y^2 = 4,
\]

and get

\[
x^2 = 4 - 4y^2 = 4(1 - y^2).
\]

Going back, we get that the volume is:

\[
\int_{-1}^{1} 4x^2 \, dy = 4 \int_{-1}^{1} 4(1 - y^2) \, dy = 16 \int_{-1}^{1} (1 - y^2) \, dy = 16(y - \frac{y^3}{3}) \bigg|_{-1}^{1}
\]

\[
= 16[(1 - \frac{13}{3}) - ((-1) - \frac{(-1)^3}{3})] = 16[\frac{2}{3} - \frac{2}{3}] = 16 \cdot \frac{4}{3} = \frac{64}{3}.
\]

**Ans.:** \( \frac{64}{3} \) (or \( 21 \frac{1}{3} \)).