Solution to a typical cross section problem

1. The base of a solid is the region inside the ellipe $x^2 + 4y^2 = 4$. Each cross-section of the solid perpendicular to the y-axis is a square. What is the volume of the solid?

Sol. of 1: The ellipse crosses the y-axis at the points $(0, 1)$ and $(0, -1)$ (set $x = 0$ in the equation $x^2 + 4y^2 = 4$ and get $4y^2 = 4$, so $y = -1, 1$).

A cross-section perpendicular to the y-axis has length $2x$ (since the distance from the y-axis of a typical point (x, y) is by definition of coordinates, x, and it goes both ways, so by symmetry it is twice x). So the area of the square above it is $(side)^2 = (2x)^2 = 4x^2$.

So the desired volume is

$$\int_{-1}^{1} 4x^2\, dy = 4\int_{-1}^{1} x^2\, dy \quad .$$

But this, right now, does not make sense. We need an integral in terms of y!

We now use the equation of the given ellipse:

$$x^2 + 4y^2 = 4 \quad ,$$

and get

$$x^2 = 4 - 4y^2 = 4(1 - y^2) \quad .$$

Going back, we get that the volume is:

$$\int_{-1}^{1} 4x^2\, dy = 4\int_{-1}^{1} 4(1 - y^2)\, dy = 16\int_{-1}^{1} (1 - y^2)\, dy = 16(y - \frac{y^3}{3})\Big|_{-1}^{1}$$

$$= 16[(1 - \frac{1^3}{3}) - ((-1) - \frac{(-1)^3}{3}] = 16[\frac{2}{3} - -\frac{2}{3}] = 16 \cdot \frac{4}{3} = \frac{64}{3} \quad .$$

Ans.: $\frac{64}{3}$ (or $21\frac{1}{3}$).