MATH 152 (07-09), Dr. Z., Solution to Second Midterm

1. (10 points) Use the integral test to determine whether the series is convergent or divergent. Explain everything!

\[ \sum_{n=1}^{\infty} \frac{n}{e^{2n}} \]

Solution of (1): The series has the same convergence status as the improper integral

\[ \int_{1}^{\infty} \frac{x}{e^{2x}} \, dx \]

So let’s try to evaluate it. First we need the indefinite integral. Writing the integral as

\[ \int_{1}^{\infty} x e^{-2x} \, dx \]

we use integration by parts with \( u = x, \ v' = e^{-2x} \). This gives \( u' = 1 \) and \( v = \frac{e^{-2x}}{-2} = (-1/2)e^{-2x}. \) So

\[
\int_{1}^{\infty} x e^{-2x} \, dx = \frac{x e^{-2x}}{-2} - \int_{1}^{\infty} (-1/2) e^{-2x} \, dx = (-1/2)xe^{-2x} + \frac{1}{2} \int e^{-2x} \, dx =
\]

\[
(-1/2)xe^{-2x} + \frac{1}{2} \frac{e^{-2x}}{-2} = (-1/2)xe^{-2x} - (1/4)e^{-2x} = -\frac{x}{2e^{2x}} - \frac{1}{4e^{2x}}
\]

Now, we put limits from 1 to \( R \):

\[
\int_{1}^{R} x e^{-2x} \, dx = -\frac{x}{2e^{2x}} - \frac{1}{4e^{2x}} \bigg|_{1}^{R} = -\frac{R}{2e^{2R}} - \frac{1}{4e^{2R}} + \frac{1}{2e^{2}} + \frac{1}{4e^{2}}
\]

Now we take the limit at \( R \to \infty \), and get

\[
\int_{1}^{\infty} x e^{-2x} \, dx = \lim_{R \to \infty} \left( -\frac{R}{2e^{2R}} - \frac{1}{4e^{2R}} + \frac{1}{2e^{2}} + \frac{1}{4e^{2}} \right)
\]

The first limit is 0 by L’Hôpital and the second is \( 1/\infty = 0 \), so the answer is a finite number, hence the improper integral converges, and by the integral test, the series converges.

Ans. to 1): Series converges by the integral test.

Important note: This is much easier with either the ratio test or root test \( (\rho = 1/e^{2} < 1) \), so if you are not specifically requested to use the integral test use another test!
2. (10 points, 5 each) Determine whether the following series converge or diverge. Explain what test(s) you are using.

(a) \( \sum_{n=1}^{\infty} \frac{7n^2 + 8n + 4\sqrt{n}}{n^4 + n + 9} \),

(b) \( \sum_{n=1}^{\infty} \frac{7 + 4\sqrt{n}}{n^{6/5}} \).

**Solution to 2a):** By the Limit Comparison Test

\[
\sum_{n=1}^{\infty} \frac{7n^2 + 8n + 4\sqrt{n}}{n^4 + n + 9}
\]

has the same convergence status as

\[
\sum_{n=1}^{\infty} \frac{7n^2}{n^4} = 7 \sum_{n=1}^{\infty} \frac{1}{n^2} ,
\]

which converges by the \( p \)-test (\( p = 2 \)).

**Ans. to 2a):** Converges by Limit Comparison and \( p \)-test.

**Solution to 2b):** By the Limit Comparison Test

\[
\sum_{n=1}^{\infty} \frac{7 + 4\sqrt{n}}{n^{6/5}}
\]

has the same convergence status as the simpler series

\[
\sum_{n=1}^{\infty} \frac{4\sqrt{n}}{n^{6/5}} = 4 \sum_{n=1}^{\infty} \frac{n^{1/2}}{n^{6/5}} = 4 \sum_{n=1}^{\infty} \frac{1}{n^{7/10}}
\]

which diverges by the \( p \)-test (\( p = 7/10 < 1 \)).

**Ans. to 2b):** Diverges by Limit Comparison and \( p \)-test.

3. (10 points: 3,3,4 resp.) Determine whether the following series converge or diverge

(a) \( \sum_{n=1}^{\infty} \frac{\sin^4 n}{n^2 + 1} \),

(b) \( \sum_{n=1}^{\infty} \frac{1}{n - 10\sqrt{n}} \),

(c) \( \sum_{n=1}^{\infty} \frac{1}{5 - 2^{-n}} \).
Solution of 3a): The trig functions sin and cos are always between $-1$ and $1$ so their square is less than 1. By the (straight) comparison test

$$\sum_{n=1}^{\infty} \frac{\sin^4 n}{n^2 + 1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

Now by Limit (or straight) comparison test, and the $p$-test ($p = 2$) this is convergent, so

Ans. to 3a): Convergent by Comparison Test and $p$-test.

Solution of 3b): You can use (Straight-) Comparison but it is much easier to use Limit-Comparison.

$$\sum_{n=1}^{\infty} \frac{1}{n - 10\sqrt{n}}$$

has the same convergence status as

$$\sum_{n=1}^{\infty} \frac{1}{n}$$,

which is the famous harmonic series (or $p$-series with $p = 1$) and so diverges.

Ans. to 3b): Divergent by Limit Comparison Test and $p$-test ($p = 1$).

Solution of 3c): By Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{1}{5 - 2^{-n}}$$

has the same convergence status as (since $2^{-n}$ is tiny in the long run)

$$\sum_{n=1}^{\infty} \frac{1}{5}$$,

and this equals $\infty$, hence diverges.

COMMON MISTAKE: Many people ‘forget’ about the 5, and usually when 5 is next to $n$ or $n^2$ they are right, but in this problem 5 is very significant since $2^{-n}$ is insignificant.

Another way: use the divergence test

$$\lim_{n \to \infty} \frac{1}{5 - 2^{-n}} = \frac{1}{5}$$

which is not 0, so it diverges by the divergence test.
Ans. to 3c): Divergent by Limit Comparison Test and common sense (or $p$-test with $p = 0$). Or: Divergent by Divergence Test.

4. (10 points) Use the sum of the first 3 terms to approximate the sum of the series. Estimate the error.

$$\sum_{n=1}^{\infty} \frac{n+1}{(n+3)5^n}.$$ 

Solution of 4: The first three terms are

$$\sum_{n=1}^{3} \frac{n+1}{(n+3)5^n} = \frac{1+1}{(1+3) \cdot 5^1} + \frac{2+1}{(2+3) \cdot 5^2} + \frac{3+1}{(3+3) \cdot 5^3} = \frac{1}{10} + \frac{3}{125} + \frac{2}{375}.$$ 

The remainder is

$$\sum_{n=4}^{\infty} \frac{n+1}{(n+3)5^n} \leq \sum_{n=4}^{\infty} \frac{1}{5^n}$$

since $(n+1)/(n+3) < 1$. The right side is a geometric series

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \frac{1}{5^4} + \frac{1}{5^5} + \frac{1}{5^6} + \ldots = \frac{1}{5^4} \cdot \left( \frac{1}{5} + \frac{1}{5^2} + \ldots \right) = \frac{1}{5^4} \cdot \frac{1}{1 - 1/5} = \frac{1}{5^4} \cdot \frac{5}{4} = \frac{1}{600}$$

(We used the geometric series sum

$$1 + r + r^2 + r^3 + \ldots = \frac{1}{1 - r}$$

).

Ans. to 4) The approximation to the series using the first three terms is $\frac{1}{10} + \frac{3}{125} + \frac{2}{375}$ and the error is $\leq 1/600$.

5. (10 points, 5 each) Determine whether the following series are absolutely convergent, conditionally convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$, 

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$.

Solution of 5a): Use the ratio test

$$a_n = \frac{n^3}{2^n},$$

4
\[ a_{n+1} = \frac{(n+1)^3}{2n+1}, \]
\[ \frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{2n+1} \cdot \frac{n^3}{2n} \]
\[ = \frac{(n+1)^32^n}{n^32^{n+1}} = \frac{(n+1)^32^n}{n^32^n \cdot 2} = \frac{(n+1)^3}{2n^3} \]

Now take the limit of the ratio
\[ \rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^3}{2n^3} = \lim_{n \to \infty} \frac{n^3}{2n^3} = \frac{1}{2}. \]

Since \( \rho = \frac{1}{2} \) is less than 1:

**Ans. to 5a):** Converges Absolutely by ratio test.

**Solution of 5b):** We first consider the absolute version
\[
\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{\ln(n+1)} = \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}
\]

By the Limit Comparison Test this has the same convergence status as
\[
\sum_{n=2}^{\infty} \frac{1}{\ln n}
, \]
and this is the same as
\[
\sum_{n=2}^{\infty} \frac{1}{n^0(\ln n)^1}
, \]
which diverges by Dr. Z’s \( p - q \) test \( (p = 0, q = 1) \). Since \( p < 1 \) it diverges. Hence the absolute version diverges, and so the series is **not** absolutely convergent.

This leaves two options: cond. conv. and divergent. Since the original series is alternating, we try to use the Alternating Series Test with
\[ b_n = \frac{1}{\ln(n+1)}, \]

This is (i) decreasing (since \( \ln(x+1) \) is increasing its reciprocal is decreasing) (ii) tends to 0 (since \( \ln(x+1) \) goes to \( \infty \) its reciprocal goes to 0) It follows by the Alternating Series Test that the series is convergent. Since it is convergent but not absolutely convergent it is **conditionally convergent**.
Ans. to 5b): Conditionally Convergent.

6. (10 points, 5 each) Determine whether the following series are absolutely convergent, conditionally convergent or divergent.

(a) \[ \sum_{n=1}^{\infty} \frac{(-3)^n n!}{n^n} \]

(b) \[ \sum_{n=1}^{\infty} \frac{2^n n^n}{n!} \]

Solution of 6a):

We use the ratio test

\[ |a_n| = \frac{3^n n!}{n^n} \]
\[ |a_{n+1}| = \frac{3^{n+1} (n+1)!}{(n+1)^{n+1}} \]

First let's form the ratio and simplify as much as (legally!) possible

\[ \frac{|a_{n+1}|}{|a_n|} = \frac{3^{n+1} (n+1)!}{(n+1)^{n+1}} \cdot \frac{3^n n!}{n^n} \]

which simplifies to

\[ \frac{3^{n+1} n (n+1)!}{3^n n! (n+1)^{n+1}} = 3 \cdot \frac{n^n (n+1)!}{(n+1)n!} \cdot \frac{n!}{(n+1)n} \]

\[ = 3 \cdot \frac{n^n (n+1)}{(n+1)^{n+1}} = 3 \cdot \frac{n^n}{(n+1)^n} \]

This is the simplified ratio. Now take the limit as \( n \to \infty \):

\[ \rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} 3 \cdot \frac{n^n}{(n+1)^n} = 3 \lim_{n \to \infty} \frac{n^n}{(n+1)^n} = \frac{3}{e} \]

Here we use the reciprocal of the famous formula in the formula sheet:

\[ \lim_{n \to \infty} \left( \frac{n+1}{n}\right)^n = e \]

(where \( e = 2.71828... \))

Since \( \rho = 3/e \) is bigger than 1 we get
Ans. to 6a): Series diverges.

**Warning:** My shortcut of “forgetting about the little ones” does not apply here.

\[
\lim_{n \to \infty} \left( \frac{n + 1}{n} \right)^3 = \lim_{n \to \infty} \left( \frac{n}{n} \right)^3 = 1
\]

and

\[
\lim_{n \to \infty} \left( \frac{n + 1}{n} \right)^{1000} = \lim_{n \to \infty} \left( \frac{n}{n} \right)^{1000} = 1
\]

But if the exponent is \( n \) then you can **NOT** do it.

**Solution of 6b):**

We use the ratio test

\[
|a_n| = \frac{2^n n^n}{n!}
\]

\[
|a_{n+1}| = \frac{2^{n+1} (n+1)^{n+1}}{(n+1)!}
\]

First let’s form the ratio and simplify as much as (legally!) possible

\[
\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1} (n+1)^{n+1}}{(n+1)!} \cdot \frac{2^n n^n}{n!}
\]

which simplifies to

\[
\frac{2^{n+1} (n+1)^{n+1}}{2^n (n+1)! n^n} = 2 \cdot \frac{(n+1)^n}{n^n}
\]

This is the **simplified ratio**. Now take the limit as \( n \to \infty \):

\[
\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} 2 \cdot \frac{(n+1)^n}{n^n} = 2e
\]

Here we use the famous formula in the formula sheet:

\[
\lim_{n \to \infty} \left( \frac{n + 1}{n} \right)^n = e
\]

(where \( e = 2.71828... \))

Since \( \rho = 2e \) is bigger than 1 we get

Ans. to 6b): Series diverges.

7. (10 points) Find the radius of convergence and interval of convergence of the series

\[
\sum_{n=1}^{\infty} \frac{(x + 1)^n}{n2^n}.
\]
Sketch of the Solution of 7: The limit of the ratios is

\[ \rho = \frac{x + 1}{2} \]

(do it!) Setting \(|\rho| < 1\) gives

\[ |\frac{x + 1}{2}| < 1 \]

which is the same as (multiply both sides by 2) as

\[ |x + 1| < 2 \]

that gives: radius of convergence is 2 and center of convergence is \(x = -1\) so the tentative interval of convergence is \((-1 - 2, -1 + 2) = (-3, 1)\).

Now we have to check the endpoints. When \(x = -3\) the power series becomes

\[ \sum_{n=1}^{\infty} \frac{(-3 + 1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \]

which is convergent by the Alternating Series Test. So we add \(x = -3\). When \(x = 1\) the power series becomes

\[ \sum_{n=1}^{\infty} \frac{(1 + 1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n} , \]

which diverges by the \(p\)-test \((p = 1)\), so we don’t add \(x = 1\) to the interval of convergence.

Ans. to 7: Radius of convergence is 2 and interval of convergence is \([-3, 1)\) that can be written as \(-3 \leq x < 1\).

8. (10 points) Find a power series representation for the function and determine the interval of convergence.

\[ f(x) = \frac{x^2}{1 + 27x^3} \]

Solution of 8:

\[ f(x) = \frac{x^2}{1 + 27x^3} = x^2 \cdot \frac{1}{1 - (-27x^3)} \]

We use the geometric series

\[ \frac{1}{1 - w} = \sum_{n=0}^{\infty} w^n , \quad (|w| < 1) \]
with \( w = -27x^3 \).

\[
f(x) = x^2 \cdot \frac{1}{1 - (-27x^3)} = x^2 \cdot \sum_{n=0}^{\infty} (-27x^3)^n = 
\]

\[
x^2 \cdot \sum_{n=0}^{\infty} (-27)^n (x^3)^n = x^2 \cdot \sum_{n=0}^{\infty} (-27)^n x^{3n} = \sum_{n=0}^{\infty} (-27)^n x^{3n+2}
\]

This is valid for \( |-27x^3| < 1 \) which is the same as \( |x|^3 < 1/27 \) which is the same as \( |x| < 1/3 \). So the interval of convergence is \((-1/3, 1/3)\).

**Ans. to 8):** The Maclaurin series of \( f(x) \) is \( \sum_{n=0}^{\infty} (-27)^n x^{3n+2} \) and the interval of convergence is \((-1/3, 1/3)\).

**VERY IMPORTANT:** Some people find the interval of convergence the long way, doing the ratio test etc. This is a **waste** of time. Whenever you use the geometric series for \( 1/(1-w) \) it is automatic that it is valid for \( |w| < 1 \) so all you have to do is solve the inequality \( |w| < 1 \), like I did above.

**9.** (10 points) Find the Maclaurin series for \( f(x) = 4e^{x+1} \) using the definition of a Maclaurin series.

**Solution of 9):**

It is tempting to first simplify

\[
f(x) = 4e^x \cdot e^1 = (4e)e^x,
\]

and then use the memorized Maclaurin series for \( e^x \) getting that the Maclaurin series for \( f(x) \) is

\[
4e \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{4e}{n!} x^n
\]

and if the question didn’t mention “from the definition” it would have been OK. But since it did we have to do it the long way. First write down \( f(x) \) and its derivatives

\[
f(x) = 4e^{x+1} \quad f'(x) = 4e^{x+1} \quad f''(x) = 4e^{x+1} \quad f'''(x) = 4e^{x+1}
\]

and you see that they are all the same, so \( f^{(n)}(x) = 4e^{x+1} \). Plugging in \( x = 0 \) we get

\[
f(0) = 4e^{0+1} = 4e \quad f'(0) = 4e^{0+1} = 4e \quad f''(0) = 4e^{0+1} = 4e \quad f'''(0) = 4e^{0+1} = 4e
\]
and in general \( f^{(n)}(0) = 4e^{0+1} = 4e \). Using the definition
\[
f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} x^n
\]
we get
\[
f(x) = \sum_{n=0}^\infty \frac{4e}{n!} x^n.
\]

Common Mistake: Many people plugged-in \( w = x + 1 \) in
\[
e^w = \sum_{n=0}^\infty \frac{w^n}{n!}
\]
getting
\[
f(x) = \sum_{n=0}^\infty \frac{4(x + 1)^n}{n!}
\]
This is the \textbf{right answer to the wrong question}, since this is a Taylor series centered at \( x = -1 \) and not a Maclaurin series (that is centered at \( x = 0 \)).

10. (10 points, 5 each) (a) Expand \( \sqrt[3]{1+x} \) as a power series. (b) Use part (a) to estimate \( \sqrt[3]{1.01} \) correct to four decimal

Solution of (a): We use the \textbf{Binomial Series} from the Formula Sheet
\[
(1 + x)^k = 1 + \sum_{n=1}^\infty \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n
\]
or in the \ldots notation
\[
(1 + x)^k = 1 + kx + \frac{k(k-1)}{2} w^x + \ldots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n + \ldots
\]
Here \( k = 1/3 \) so

Ans. to a)
\[
(1 + x)^{1/3} = 1 + \sum_{n=1}^\infty \frac{(1/3)(1/3-1)(1/3-2)\cdots(1/3-n+1)}{n!} x^n,
\]
or even more compactly
\[
(1 + x)^{1/3} = \sum_{n=0}^\infty \binom{1/3}{n} x^n.
\]
Solution of b): We plug-in $x = .01$ and $k = 1/3$ into the ... form and write down the first few terms

$$(1+0.01)^{1/3} = 1+(1/3)(0.01)+\frac{(1/3)(1/3 - 1)}{2}(0.01)^2 + \frac{(1/3)(1/3 - 1)(1/3 - 2)}{6}(0.01)^3 + \ldots.$$ 

This is an alternating series and we quit as soon as the terms are less the prescribed error (four decimals means that the error is $5 \cdot 10^{-5}$). Already the third term is less than that, so we only keep the first two terms and the approximation is

$$(1 + 0.01)^{1/3} \approx 1 + (1/3)(0.01) = 1.0033$$

Ans. to 10b): 1.0033.