Dr. Z's Math152 Handout #9.3 [Separable Equations]

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Problem Type 9.3a: Solve the differential equation

$$y' = \frac{A(x)}{B(y)} \quad or \quad y' = A(x)B(y) \quad etc.$$

Example Problem 9.3a: Solve the differential equation

$$y' = y^2 \sec x$$

Steps	Example

Steps

1. If not already written like this, replace y' by $\frac{dy}{dx}$. Treat dy and dx as algebraic quantities and **separate** the x part from the y part.

$$\frac{dy}{dx} = \frac{A(x)}{B(y)} \quad or \quad \frac{dy}{dx} = A(x)B(y) \quad etc.$$

implies

$$B(y)dy = A(x)dx \quad respectively$$

$$\frac{dy}{B(y)} = A(x)dx \quad etc.$$

Example

1.

$$\frac{dy}{dx} = y^2 \sec x$$

implies

$$\frac{dy}{y^2} = \sec x \ dx$$

which is the same as

$$y^{-2} \ dy = \sec x \ dx$$

2. Apply the Integral sign to both sides, and perform the integration. Only put the $+C$ on one side.

2.

$$\int y^{-2} \ dy = \int \sec x \ dx \quad ,$$

gives

$$\frac{-1}{y} = \ln|\sec x + \tan x| + C$$

3. If possible, solve for y. Otherwise leave it in implicit form. If there is an initial condition then plug it in and solve for C. If nothing is mentioned (like in this problem), then leave C alone.

3.

$$y = \frac{-1}{\ln|\sec x + \tan x| + C}$$

Ans.: $y = \frac{-1}{\ln|\sec x+\tan x|+C}$.

Problem Type 9.3b: Find an eqation of the curve that passes through the point (a, b) and whose slope at (x, y) is $A(y)/B(x)$.

Example Problem 9.3b: Find an eqation of the curve that passes through the point $(1, 1)$ and whose slope at (x, y) is y^2/x^3.

Steps	**Example**
1. Slope is derivative, so set it equal to $\frac{dy}{dx}$. Treat dy and dx as algebraic quantities and **separate** the x part from the y part. $$\frac{dy}{dx} = \frac{A(y)}{B(x)}$$ implies $$\frac{dy}{A(y)} = \frac{dx}{B(x)}$$	**1.** $$\frac{dy}{dx} = \frac{y^2}{x^3}$$ implies $$\frac{dy}{y^2} = \frac{dx}{x^3}$$ which is the same as $$y^{-2}\ dy = x^{-3}\ dx$$
2. Apply the Integral sign to both sides, and perform the integration. Only put the $+C$ on one side.	**2.** $$\frac{y^{-1}}{-1} = \frac{x^{-2}}{-2} + C$$ which gives $$\frac{-1}{y} = \frac{-1}{2x^2} + C$$

3. Plug in the point $(x = a, y = b)$ and solve for C. Plug back that value for C and try to express y in terms of x if possible. Otherwise leave it in implicit form.

3.

$$\frac{-1}{1} = \frac{-1}{2 \cdot 1^2} + C$$

giving $C = -1/2$. Incorporating that C gives

$$\frac{-1}{y} = \frac{-1}{2x^2} - \frac{1}{2}$$

and algebra gives

$$y = \frac{2x^2}{1 + x^2} \quad .$$

Ans.: $\frac{2x^2}{1+x^2}$.