Dr. Z's Math152 Handout #7.8 [Improper Integrals]

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Problem Type 7.8a: Determine whether the following integral is convergent or divergent. Evaluate it if it is convegent.

$$\int_a^{\infty} f(x)\, dx \quad ,$$

where $f(x)$ is 'easy' to integrate.

Example Problem 7.8a: Determine whether the following integral is convergent or divergent. Evaluate it, if it is convegent.

$$\int_1^{\infty} \frac{1}{(4x+1)^2}\, dx$$

Steps

1. Replace the ∞ by R in the integral, and evalute it, pretending that R is a number.

$$\int_a^R f(x)\, dx$$

2. You should get an expression in R. Take the limit as $R \to \infty$, using what you know from Calc I (including L'Hôpital's rule if you wish). If the limit exists, then the improper integral **coneverges** , and the value of the limit is the value of the improper integral. If the limit does not exist (i.e. is divergent) (either it goes to infinity or it oscillates), then the improper integral is **divergent**.

Example

1.: We have to evaluate

$$\int_1^R \frac{1}{(4x+1)^2}\, dx \quad .$$

using the substitution $u = 4x+1$ we have $dx = du/4$ and the limits of integration are $u = 5$ and $u = 4R+1$, so we have

$$\int_1^R \frac{1}{(4x+1)^2}\, dx = \frac{1}{4}\int_5^{4R+1} \frac{1}{u^2}\, du = \frac{-1}{4u}\Big|_5^{4R+1}$$

$$= \frac{1}{4}\left(\frac{1}{5} - \frac{1}{4R+1}\right)$$

2.

$$\lim_{R \to \infty} \frac{1}{4}\left(\frac{1}{5} - \frac{1}{4R+1}\right) = \frac{1}{20} - \lim_{R \to \infty} \frac{-1}{4(4R+1)} = \frac{1}{20} \quad .$$

Ans.: The improper integral converges and it is equal to $1/20$.

Problem Type 7.8b: Determine whether the following integral is convergent or divergent. Evaluate it if it is convegent.

$$\int_a^b f(x)\, dx \quad ,$$

where $f(x)$ "blows up" either at $x = a$ or $x = b$ or both.

Example Problem 7.8b: Determine whether the following integral is convergent or divergent. Evaluate it, if it is convegent.

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx$$

Steps

1. Decide which of the two ends it blows up at (if it blows up at both, it is best to split the integral into two parts, and treat each of them separately). Replace the 'bad endpoint' by the variable t. Then evaluate the integral pretending that t is a number. Let's say, for the sake of example, like in this problem) that $f(x)$ blows up at $x = a$.

$$\int_t^b f(x)\, dx$$

2. You should get an expression in t. Take the limit (from the right, if the blow-up point is the a, from the left, if the blow-up point is the b of that expression, using what you know from Calc I (including L'Hôpital's rule if you wish). If the limit exists, then the improper integral **coneverges** , and the value of the limit is the value of the improper integral. If the limit does not exist (i.e. is divergent) (either it goes to infinity or it oscillates), then the improper integral is **divergent**.

Example

1.: We have to evaluate

$$\int_t^1 x^{-1/2}\, dx \quad .$$

This equals

$$\frac{x^{1/2}}{1/2}\Big|_t^1 = 2\sqrt{x}\Big|_t^1 = 2(1 - \sqrt{t})$$

2.

$$\lim_{t \to 0^+} 2(1 - \sqrt{t}) = 2(1 - \sqrt{0}) = 2 \quad .$$

Ans.: The improper integral converges and it is equal to 2.

Problem Type 7.8c: Determine whether the following integral is convergent or divergent. Evaluate it if it is convegent.

$$\int_a^b f(x)\, dx \quad ,$$

where $f(x)$ "blows up" either at $x = a$ or $x = b$ or both.

Example Problem 7.8c: Determine whether the following integral is convergent or divergent. Evaluate it, if it is convegent.

$$\int_0^1 \frac{1}{x}\, dx$$

Steps

1. Decide which of the two ends it blows up at (if it blows up at both, it is best to split the integral into two parts, and treat each of them separately). Replace the 'bad endpoint' by the variable t. Then evaluate the integral pretending that t is a number. Let's say, for the sake of example, like in this problem) that $f(x)$ blows up at $x = a$.

$$\int_t^b f(x)\, dx$$

2. You should get an expression in t. Take the limit (from the right, if the blow-up point is the a, from the left, if the blow-up point is the b of that expression, using what you know from Calc I (including L'Hôpital's rule if you wish). If the limit exists, then the improper integral **conev-erges** , and the value of the limit is the value of the improper integral. If the limit does not exist (i.e. is divergent) (either it goes to infinity or it oscillates), then the improper integral is **divergent**.

Example

1.: We have to evaluate

$$\int_t^1 \frac{1}{x}\, dx \quad .$$

This equals

$$\ln x|_t^1 = \ln 1 - \ln t = 0 = -\ln t$$

2.

$$\lim_{t \to 0^+} -\ln t = \infty$$

Ans.: The improper integral diverges.

Problem Type 7.8d: Use the comparison theorem to determine whether the integral is convergent or divergent.

$$\int_a^\infty f(x)\, dx \quad ,$$

Example Problem 7.8d: Determine whether the following integral is convergent or divergent. Do not evaluate!

$$\int_1^\infty \frac{\cos^2 x}{1+x^2}\, dx$$

Steps

1. Decide on an "easy" integrand to compare it with, hoping that the rest is bounded by some number. Prove that the "easy" version is convergent.

2. Look at the rest of the integrand and show that it is smaller than some fixed number.

Example

1.: $\frac{\cos^2 x}{1+x^2}$ is 'similar' to $\frac{1}{1+x^2}$ is 'similar' to which we know (or can find out) is convergent. Indeed

$$\int_0^\infty \frac{1}{1+x^2}\, dx = \tan^{-1} x|_0^\infty = \tan^{-1}\infty - \tan^{-1} 1$$

$$= \pi/2 - \pi/4 = \pi/4 \quad .$$

2. $\cos x$ is *always* between -1 and 1 so $\cos^2 x$ is always smaller than 1. Hence

$$\frac{\cos^2 x}{1+x^2} \le \frac{1}{1+x^2} \quad .$$

So we have

$$\int_1^\infty \frac{\cos^2 x}{1+x^2} \le \int_1^\infty \frac{1}{1+x^2} \quad .$$

Since $\int_1^\infty \frac{1}{1+x^2} < \infty$, from Step 1, it follows that $\int_1^\infty \frac{\cos^2 x}{1+x^2} < \infty$, so the integral converges.

Ans.: The improper integral converges.

Note: we could have used the comparison test to prove that $\int_1^\infty \frac{1}{1+x^2}$ is convergent (if we forgot how to integrate it using $\tan^{-1}$) by using the comparison

$$\frac{1}{1+x^2} < \frac{1}{x^2}$$

for $x \ge 1$, and use the fact that $\int_1^\infty \frac{1}{x^2}$ is convergent.

Important fact for future reference: The improper integral $\int_1^\infty \frac{1}{x^p}$ is convergent for $p > 1$ and divergent for $p \le 1$.