Dr. Z's Math152 Handout #7.3 [Trigonometric Substitution]

By Doron Zeilberger

Problem Type 7.3a: Integrate expressions involving $\sqrt{a^2-x^2}$, or $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$, where a is some number.

Example Problem 7.3a: Evaluate the integral

$$\int \sqrt{1-x^2}\, dx$$

Steps

1. If $\sqrt{a^2-x^2}$ shows up use the substitution $x = a\sin\theta$ and you'll have to use

$$1-\sin^2\theta = \cos^2\theta \quad .$$

If $\sqrt{a^2+x^2}$ shows up use the substitution $x = a\tan\theta$, and you'll have to use

$$1+\tan^2\theta = \sec^2\theta \quad .$$

If $\sqrt{x^2-a^2}$ shows up use the substitution $x = a\sec\theta$. and you'll have to use

$$\sec^2\theta - 1 = \tan^2\theta \quad .$$

Get some trig integral of the type of section 7.2.

Example

1. Use the substitution $x = \sin\theta$, implying $dx = \cos\theta\ d\theta$.

$$\int \sqrt{1-x^2}\ dx = \int \sqrt{1-\sin^2\theta}\ \cos\theta\ d\theta$$

$$= \int \sqrt{\cos^2\theta}\cos\theta\ d\theta = \int \cos\theta\ \cos\theta\ d\theta = \int \cos^2\theta\ d\theta \quad .$$

Steps

2. Use the know-how of Section 7.2 (see Handout 7.2), to evaluate this trig integral.

Example

2.

$$\int \cos^2\theta\ d\theta = \int \frac{1+\cos 2\theta}{2}\ d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} \quad .$$

3. Convert the answer back to the x language.

3. Since $x = \sin\theta$, we have $\theta = \sin^{-1} x$, so a correct, but simplistic answer is

$$(\sin^{-1} x)/2 + (\sin 2(\sin^{-1} x))/4 \quad .$$

But it is better to convert the $\sin 2\theta$ into $2\sin\theta\cos\theta$ giving

$$(\sin 2\theta)/4 = (1/2)\sin\theta\cos\theta = (1/2)(x)\sqrt{1-x^2} \quad .$$

Ans.: $\frac{1}{2}(\sin^{-1} x) + \frac{1}{2}x\sqrt{1-x^2}$.