Dr. Z's Math152 Handout #7.2 [Trigonometric Integrals]

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Problem Type 7.2a: Integrate an **odd** power of a sine or cosine.

$$\int \sin^{2n+1} x \, dx \quad OR \quad \int \cos^{2n+1} x \, dx \quad (n = 1, 2, \ldots)$$

Example Problem 7.2a: Evaluate the integral

$$\int \sin^3 x \, dx$$

Steps

1. Rewrite $\sin^{2n+1} x \, dx$ as $(-\sin^{2n} x)(-\sin x \, dx)$, and $\cos^{2n+1} x \, dx$ as $(\cos^{2n} x)(\cos x \, dx)$. Get ready to make the **substition** $u =$ TheOtherGuy (i.e. if you have an odd power of sine, then $u = \cos x$ and if you have an odd power of cosine, then $u = \sin x$.)

Example

1.

$$\int \sin^3 x \, dx = \int (-\sin^2 x)(-\sin x \, dx) \quad .$$

Get ready to make the substitution $u = \cos x$, that implies $du = (-\sin x)dx$ since $(\cos x)' = -\sin x$.

2. Use the famous trig-identity $\sin^2 x + \cos^2 = 1$ in one of the equivalent forms $\sin^2 x = 1 - \cos^2 x$, $\cos^2 x = 1 - \sin^2 x$, to have everything in terms of the other guy, also replacing $(-\sin x \, dx)$ by $d(\cos x)$ or $(\cos x \, dx)$ by $d(\sin x)$, as the case may be. Then make the substitution. $u = \cos x$ or $u = \sin x$ (as the case may be).

$$\int (-\sin^{2n} x)(-\sin x \, dx) = \int (-\sin^{2n} x) \, d(\cos x)$$

$$= \int -(\sin^2 x)^n \, d(\cos x)$$

$$= \int -(1 - \cos^2 x)^n \, d(\cos x) = \int -(1 - u^2)^n \, du$$

and analogously for the other case.

2. $u = \cos x$, so

$$\int \sin^3 x \, dx = \int (-\sin^2 x)(-\sin x \, dx) =$$

$$= \int -\sin^2 x \, d(\cos x) = \int -(1 - \cos^2 x) d(\cos x)$$

$$= \int -(1 - u^2) \, du$$

3. Evaluate the u-integral by first using algebra to simplify the integrand (if necessary). Then replace u by $\cos x$ or $\sin x$ (as the case may be). Add $+C$ at the very end.

3.

$$= \int -(1-u^2)\ du = -u+u^3/3 = -\cos x + \frac{\cos^3 x}{3} + C$$

Ans: $-\cos x + \frac{\cos^3 x}{3} + C$.

Problem Type 7.2b:

Integrate a product of sine- and cosine-powers where at least one of the powers is odd

$$\int \cos^{whatever} x\ \sin^{2n+1} x\ dx \quad OR \quad \int \sin^{whatever} x\ \cos^{2n+1} x\ dx \quad (n = 1, 2, \ldots)$$

Example Problem 7.2b:

$$\int \cos^2 x\ \sin^5 x\ dx$$

Steps

1. Locate the trig-function that has the odd-power. Make the **substition** $u =$ TheOtherGuy. If they both have odd powers, then go by the lower one. If they are identical then it does not matter which one you pick. Otherwise do the same as in 7.2a. Make sure you convert *everything* into the u-language.

2. Use the famous trig-identity $\sin^2 x + \cos^2 = 1$ in one of the equivalent forms $\sin^2 x = 1-\cos^2 x$, $\cos^2 x = 1-\sin^2 x$, to have everything in terms of the other guy, also replacing $(-\sin x\ dx)$ by $d(\cos x)$ or $(\cos x\ dx)$ by $d(\sin x)$, as the case may be. Then make the substitution $u = \cos x$ or $u = \sin x$ (as the case may be).

Example

1.

$$\int \cos^2 x \sin^5 x\ dx = \int -(\cos^2 x)(\sin^2 x)^2(-\sin x\ dx) \quad .$$

Get ready to make the substitution $u = \cos x$, that implies $du = (-\sin x)dx$ since $(\cos x)' = -\sin x$.

2. $u = \cos x$, so

$$= \int -(\cos^2 x)(1-\cos^2 x)^2\ d(\cos x)$$

$$= \int -u^2(1-u^2)^2\ du$$

3. Evaluate the u-integral by first using algebra to simplify the integrand (if necessary). Then replace u by $\cos x$ or $\sin x$ (as the case may be). Add $+C$ at the very end.

3.

$$= \int -u^2(1-2u^2+u^4)\ du = \int (-u^2+2u^4-u^6)\ du =$$

$$-u^3/3+2u^5/5-u^7/7 = -\frac{\cos^3 x}{3}+\frac{2\cos^5 x}{5}-\frac{\cos^7 x}{7}+C$$

Ans: $-\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C.$

Problem Type 7.2c: Integrate an even power of sine or cosine

$$\int \sin^{2n} x\ dx \quad OR \quad \int \cos^{2n} x\ dx \quad (n = 1, 2, \ldots)$$

Example Problem 7.2c: Evaluate the integral

$$\int \cos^4 x\ dx$$

Steps

1. Use the trig-identities

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

possibly more than once, with θ either x, $2x$ or whatever shows up.

Example

1. Focus on the trig (and algebra)! There is no integration or calculus at this step.

$$\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{(1+\cos 2x)^2}{4} =$$

$$\frac{1+2\cos 2x+\cos^2 2x}{4} = \frac{1}{4}+\frac{2\cos 2x}{4}+\frac{\cos^2 2x}{4} =$$

$$\frac{1}{4}+\frac{\cos 2x}{2}+\frac{(1+\cos 4x)/2}{4} = \frac{3}{8}+\frac{\cos 2x}{2}+\frac{\cos 4x}{8}$$

2. Integrate the simplified trig-expression using repeatedly $\int \sin Ax\ dx = -(\cos Ax)/A$ and/or $\int \cos Ax\ dx = (\sin Ax)/A$. Add $+C$ at the very end.

2.

$$\int \cos^4 x\ dx = \left(\frac{3}{8}+\frac{\cos 2x}{2}+\frac{\cos 4x}{8}\right)\ dx$$

$$\frac{3x}{8}+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+C$$

Ans.: $\frac{1}{4}(\frac{3x}{2}+\sin 2x+\frac{\sin 4x}{8})+C$

Powers of tangents: $\int \tan x\ dx = \ln|\sec x| + C$. To do $\int \tan^2 x\ dx$ you use the trig-identity $\tan^2 x = \sec^2 x - 1$ so $\int \tan^2 x\ dx = \int (\sec^2 x - 1)\ dx = \tan x - x + C$. To do higher powers of $\tan x$, you separate $\tan^2 x$ and relpace it by $\sec^2 x - 1$, and then use, if neccessary the substitution $u = \tan x$. e.g.

$$\int \tan^4 x\ dx = \int (\tan^2 x)(\tan^2 x)\ dx = \int (\tan^2 x)(\sec^2 x - 1)\ dx =$$

$$\int \tan^2 x\ \sec^2 x\ dx - \int \tan^2 x\ dx.$$

To do $\int (\tan^2 x)(\sec^2 x)\ dx$ use $u = \tan x$, giving $du = (\sec^2 x)\ dx$, making it $\int u^2\ du = u^3/3 = (\tan^3 x)/3$, and $\int \tan^2 x\ dx$ we did above. So $\int \tan^4 x\ dx = (\tan^3 x)/3 - \tan x + x + C$.

Mixtures of powers of tan and sec: If the power of sec is even use $u = \tan x$, if the power of tan is odd, use $u = \sec x$. If the power of sec is odd and the power of tan is even, bad luck! Use trig-identities and/or integration-by-parts.