Dr. Z's Math152 Handout #11.9 [Representations of Functions as Power Series]

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Problem Type 11.9a: Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{x^M}{a + bx^N} \quad ,$$

for integers N and M and numbers a and b.

Example Problem 11.9a: Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{x^3}{4 + 36x^2} \quad .$$

Steps

1. You'd like to use the famous geometrical series power series, and let's use the letter z:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

whose radius of convergence is 1, i.e. it is valid for $|z| < 1$.

With this in mind we rewrite our function of x, $f(x)$, as

$$x^M \cdot \frac{1}{a + bx^N} = x^M \cdot \frac{1}{a(1 + (b/a)x^N)} = \frac{x^M}{a} \cdot \frac{1}{1 - (-b/a)x^N} \quad .$$

Example

1.

$$\frac{x^3}{4 + 36x^2} = x^3 \cdot \frac{1}{4 + 36x^2} = x^3 \cdot \frac{1}{4(1 + 9x^2)}$$

$$= \frac{x^3}{4} \cdot \frac{1}{1 - (-9x^2)} \quad .$$

2. Plug-in $z = (-b/a)x^N$ into the formula

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (valid\ for\ |z| < 1)$$

to get

$$\frac{1}{1-(-b/a)x^N} = \sum_{n=0}^{\infty}((-b/a)x^N)^n$$

$$(valid\ for\ |(-b/a)x^N| < 1)$$

Simplify!

2. Plugging $z = -9x^2$ into

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

(valid for $z| < 1$) gives

$$\frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty}(-9x^2)^n \quad ,$$

valid for $|9x^2| < 1$, which simplifies to

$$\frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty}(-9)^n x^{2n}$$

(valid for $|x^2| < 1/9$) and hence

$$\frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty}(-9)^n x^{2n}$$

(valid for $|x| < 1/3$).

3. Finish it up by multiplying both sides by $\frac{x^M}{a}$, and simplifying. Also simplify the condition of validity to get the interval of convergence.

3.

$$\frac{x^3}{4+36x^2} = \frac{x^3}{4} \cdot \frac{1}{1-(-9x^2)} =$$

$$\frac{x^3}{4}\sum_{n=0}^{\infty}(-9)^n x^{2n} = \sum_{n=0}^{\infty}\frac{(-9)^n}{4}x^{2n+3} =$$

$(1/4)x^3+(-9/4)x^5+(81/4)x^7+\ldots (valid\ for\ |x| < 1/3)$

Now $|x| < 1/3$ is the same as the interval $(-1/3, 1/3)$.

Ans.: The power-series representation is

$$\frac{x^3}{4+36x^2} = \sum_{n=0}^{\infty}\frac{(-9)^n}{4}x^{2n+3} \quad ,$$

and the interval of convergence is $(-1/3, 1/3)$. (the radius of convergence is $1/3$).

Problem Type 11.9b: Evaluate the indefinite integral as a power series. What is the radius of convergence?

$$\int f(x)\ dx$$

where $f(x)$ is a function whose power-series representation you can find out (either from the formula sheet or by manipulating geometric series like in 11.9a).

Example Problem 11.9b: Evaluate the indefinite integral as a power series. What is the radius of convergence?

$$\int \frac{x^3}{4+36x^2}\ dx$$

Steps	**Example**
1. Express the integrand as a power-series. In other words, do 11.9a.	**1.** Doing 11.9a we have $$\frac{x^3}{4+36x^2} = \sum_{n=0}^{\infty} \frac{(-9)^n}{4} x^{2n+3} \quad ,$$
2. Integrate term-by-term, using the famous formula $$\int x^m\ dx = \frac{x^{m+1}}{m+1}$$ Do not worry about the $+C$ until the very end.	**2.** $$\int \frac{x^3}{4+36x^2}\ dx$$ $$= \sum_{n=0}^{\infty} \frac{(-9)^n}{4} \int x^{2n+3}\ dx \quad ,$$ $$= \sum_{n=0}^{\infty} \frac{(-9)^n}{4} \frac{x^{2n+4}}{2n+4}$$ $$= \sum_{n=0}^{\infty} \frac{(-9)^n}{4(2n+4)} x^{2n+4} \quad .$$
3. Add $+C$ at the **beginning**, and note that the radius of convergence is **always** the same as that of the integrand. We found out in 11.9a that it was 1/3, so:	**3. Ans.:** $$C + \sum_{n=0}^{\infty} \frac{(-9)^n}{4(2n+4)} x^{2n+4} \quad ,$$ and the radius of convergence is 1/3.