Dr. Z's Math152 Handout #11.8 [Power Series]

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Problem Type 11.8a: Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} b_n (x+c)^n \quad ,$$

for some sequence b_n and specific number c. (x is a generl symbol).

Example Problem 11.8a: Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n(x+4)^n}{4^{n+1}} \quad .$$

Steps

1. Try to use the ratio tests. First try to evaluate the limit of the ratios

$$\lim_{n \to \infty} \frac{b_{n+1}(x+c)^{n+1}}{b_n (x+c)^n} = (x+c) \lim_{n \to \infty} \frac{b_{n+1}}{b_n} \quad .$$

That limit should be some simple expression in x.

2. Take that expression let's call it $LimRatio(x)$, and solve, for x:

$$|LimRatio(x)| < 1.$$

Rewrite it in the form $|x-c| < R$. The c is the **center of convergence** (very often 0), and the R is called the **radius of convergence**. The tentative interval of convergence is $(c-R, c+R)$. Later on we will have to decide which, if any, of the two endpoints ($x = c-R$ and $x = c+R$) to add to that interval.

Example

1.

$$\lim_{n \to \infty} \frac{\frac{(n+1)(x+4)^{n+1}}{4^{n+2}}}{\frac{n(x+4)^n}{4^{n+1}}}$$

$$\lim_{n \to \infty} \frac{(n+1)(x+4)}{4n} = \frac{x+4}{4} \lim_{n \to \infty} \frac{n+1}{n}$$

$$= \frac{x+4}{4}$$

2.

$$|\frac{x+4}{4}| < 1$$

is the same as $|x+4| < 4$, and writing it $|x-(-4)| < 4$, we see that the center of convergence is $x = -4$, and the radius of convergence is 4. The tentative interval of convergence is $(-8, 0)$.

3. For each of the two endpoints of the interval of convergence, $x = c - R$ and $x = c + R$, plug them in, and decide, on their own merit, whether the resulting series converge or diverge. If either (or both) converge, change the tentative interval of convergence, that was open, into either half-open, or closed interval.

3. Plugging $x = -8$ into our series, gives the series

$$\sum_{n=1}^{\infty} \frac{n(-8+4)^n}{4^{n+1}} = \sum_{n=1}^{\infty} \frac{n(-4)^n}{4^{n+1}} = \frac{1}{4} \sum_{n=1}^{\infty} (-1)^n n \quad ,$$

which diverges by the divergence test. So $x = -8$ didn't make it.

Plugging $x = 0$ gives:

$$\sum_{n=1}^{\infty} \frac{n(0+4)^n}{4^{n+1}} = \frac{1}{4} \sum_{n=1}^{\infty} n \quad ,$$

that also diverges. So $x = 0$ didn't make it either. The tentative interval of convergence $(-8, 0)$ stayed as it is.

Final answer: Radius of convergence is 4; The interval of convergence is $(-8, 0)$ (that can be written $\{x; -8 < x < 0\}$).

Problem Type 11.8a': Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} b_n (x+c)^n \quad ,$$

for some sequence b_n and specific number c. (x is a generl symbol).

Example Problem 11.8a': Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n}\, 3^{n+1}} \quad .$$

Steps	**Example**

1. Try to use the ratio tests. First try to evaluate the limit of the ratios

$$\lim_{n\to\infty} \frac{b_{n+1}(x+c)^{n+1}}{b_n (x+c)^n} = (x+c) \lim_{n\to\infty} \frac{b_{n+1}}{b_n} \quad .$$

That limit should be some simple expression in x.

1.

$$\lim_{n\to\infty} \frac{\frac{(x-2)^{n+1}}{\sqrt{n+1}\, 3^{n+2}}}{\frac{(x-2)^n}{\sqrt{n}\, 3^{n+1}}}$$

$$\lim_{n\to\infty} \frac{(x-2)\sqrt{n}}{3\sqrt{n+1}} = \frac{x-2}{3} \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{1/2}$$

$$= \frac{x-2}{3}$$

2. Take that expression let's call it $LimRatio(x)$, and solve, for x:

$$|LimRatio(x)| < 1.$$

Rewrite it in the form $|x - c| < R$. The c is the **center of convergence** (very often 0), and the R is called the **radius of convergence**. The tentative interval of convergence is $(c-R, c+R)$. Later on we will have to decide which, if any, of the two endpoints ($x = c - R$ and $x = c + R$) to add to that interval.

2.

$$|\frac{x-2}{3}| < 1$$

is the same as $|x - 2| < 3$ and we see that the center of convergence is $x = 2$, and the radius of convergence is 3. The tentative interval of convergence is $(-1, 5)$.

3. For each of the two endpoints of the interval of convergence, $x = c - R$ and $x = c + R$, plug them in, and decide, on their own merit, whether the resulting series converge or diverge. If either (or both) converge, change the tentative interval of convergence, that was open, into either half-open, or closed interval.

3. Plugging $x = -1$ into our series, gives the series

$$\sum_{n=1}^{\infty} \frac{(-1-2)^n}{\sqrt{n}3^{n+1}} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad ,$$

which converges by the alternating series test. So we have to add $x = -1$ into the tentative interval of convergence.

Plugging $x = 5$ into our series, gives the series

$$\sum_{n=1}^{\infty} \frac{(5-2)^n}{\sqrt{n}3^{n+1}} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad ,$$

which diverges by the p-test. So we don't include $x = 5$.

Final answer: Radius of convergence is 3; The interval of convergence is $[-1, 5)$ (that can be written $\{x; -1 \leq x < 5\}$).