

## Dr. Z's Math152 Handout #11.6 [Absolute Convergence and the Ratio and Root Tests]

By Doron Zeilberger

**Problem Type 11.6a:** Determine whether the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, conditionally convergent, or divergent.

**Example Problem 11.6a:** Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

$$(a) \quad \sum_{n=1}^{\infty} \frac{n^2}{(-2)^n} \quad ,$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \quad ,$$

$$(c) \quad \sum_{n=1}^{\infty} \frac{(-2)^n n!}{n^n} \quad .$$

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### Steps

1. Test for **absolute convergence** by testing whether the series  $\sum_{n=1}^{\infty} |a_n|$  is convergent. Use any test, but the most efficient and popular ones are the **ratio test** that says that if  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n|$  is  $< 1$  then it is abs. convergent, whereas if it is  $> 1$  then it is divergent, and if it equals 1 exactly, then there is nothing you can say. The ratio test is analogous but with  $\lim_{n \rightarrow \infty} |a_n|^{1/n}$  instead. If you have  $n!$ , or polynomials, then it is easier to use the ratio test. If you have terms like  $c^n$ , where  $c$  is a constant, then you can use either tests, and if you have terms like  $(3n+5)^n/(5n+7)^n$  (but no  $n!$ ), then it is much easier to use the root test.

### Example

1. For (a) we use  $a_n = \frac{n^2}{(-2)^n}$ , so

$$|a_{n+1}/a_n| = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{(n+1)^2}{2n^2}$$

whose limit is  $1/2$ . Since this is  $< 1$ , we are done!

**Ans. to (a):** absolutely convergent.

For (b) we use either test.  $a_n = (-1)^{n+1} n^{-1/2}$ , so  $|a_{n+1}/a_n| = (n/(n+1))^{1/2}$ , whose limit is 1. Alternatively,  $|a_n|^{1/n} = 1/\sqrt[n]{n}$ , whose limit is also 1, so either way the test is inconclusive. On the other hand  $\sum_{n=1}^{\infty} |a_n|$  is a p-series with  $p = 1/2$ , so we know by the p-test that it is divergent. Hence the series  $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-1/2}$  is **definitely** not abs. conv. Stand by for the next step when we decide whether it is cond. conv. or outright divergent.

For (c)  $a_n = \frac{(-2)^n n!}{n^n}$ , and the  $n!$  calls for the ratio test.

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}} \\ &= \frac{2(n+1)n^n}{(n+1)^{n+1}} = \frac{2n^n}{(n+1)^n} = 2 \left( \frac{n}{n+1} \right)^n\end{aligned}$$

This brings to mind the famous limit

$$\lim_{n \rightarrow \infty} ((n+1)/n)^n = e$$

(it is in the formula sheet). So, for this problem,

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} 2 \left( \frac{n}{n+1} \right)^n = \lim_{n \rightarrow \infty} 2 / \left( \frac{n+1}{n} \right)^n = 2/e.$$

Since  $e = 2.718\dots$ ,  $2/e$  is  $< 1$  and it follows by the ratio test that we are done, and:

**Ans. to (c):** The series is abs. conv.

**2.** If we made it to this step, it means that the series is not abs. conv. Use the ping-pong test, the p-test, the integral test, or whatever to decide whether the series is convergent or divergent.

**2.** Only (b) survived to this step, since we know that the series is not abs. conv. But it is an alternating series, that is convergent by the ping-pong test. A series that is not abs. conv. but nevertheless conv. is called **conditionally convergent**.

**Ans. to (b):** The series is cond. conv.