Dr. Z's Math152 Handout #11.2 [Series]

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Problem Type 11.2a: Determine whether the series is convergent or divergent if it is convergent, find its sum

$$\sum_{n=1}^{\infty} a_n \quad ,$$

where $\lim a_n$ does not exist or exists but is not zero.

Example Problem 11.2a: For each of the two series below, determine whether they converge or diverge .

$$(a) \ \sum_{n=1}^{\infty} (-1)^n \quad , \quad (b) \ \sum_{n=1}^{\infty} \frac{n}{n+6} \quad ,$$

Steps	**Example**
1. Always try to be pessimistic and try to prove divergence, using the **divergence test** that says that if $\lim_{n\to\infty} a_n$ does not exist (i.e. diverges qua **sequence**) or even if does exist, but is not zero, then the **series** automatically is divergent.	**1.** $\lim_{n\to\infty}(-1)^n$ does not exist since $\{1,-1,1,-1,1,-1\ldots,\}$ can never make up its mind, so **Ans. to (a)**: divergent by the divergence test (limit of sequence does not exist)
Warning: The converse is not always true. It may happen that $\lim_{n\to\infty} a_n = 0$ but the series is not convergent. e.g. $\sum_{n=1}^{\infty} 1/n$ that is the famous harmonic series that diverges, but $\lim_{n\to\infty} 1/n$ equals 0.	$$\lim_{n\to\infty} \frac{n}{n+6} = \lim_{x\to\infty} \frac{x}{x+6} = 1 \quad ,$$ exists (it equals 1). Since it is not 0, the series diverges. **Ans. to (b)**: divergent by the divergence test (limit of sequence is not equal to 0).

Problem Type 11.2b: Determine whether the series is convergent or divergent if it is convergent, find the sum

$$\sum_{n=Start}^{\infty} Ar^n \quad ,$$

(for some numbers A and r).

Example Problem 11.2b: For each of the two series below, determine whether they converge or diverge. If they converge, finds its sum

$$(a) \ \sum_{n=2}^{\infty} 2\frac{(-3)^n}{5^n} \quad , \quad (b) \ \sum_{n=1}^{\infty} \frac{7^n}{6^n} \quad , \quad .$$

Steps	Example
1. These are **geometrical series** where the ratio of consecutive terms is **constant**. Find the **first term** by plugging-in $n = Start$ into the term, and call it a, then find r by taking ratios or simply by inspection, (what's inside $()^n$).	**1.** (a) Plugging $n = 2$ into $2\frac{(-3)^n}{5^n}$ gives $$a = 2\frac{(-3)^2}{5^2} = \frac{18}{25} \quad .$$ The ratio, r, is obviously $-3/5$. So for (a), $a = 18/25, r = -3/5$. For (b) $a = 7/6$ and $r = 7/6$.
2. If r is (strictly!) between -1 or 1 (i.e. $\lvert r\rvert < 1$), then the series is **convergent** and equals $$\frac{a}{1-r} \quad .$$ Otherwise (i.e. $\lvert r\rvert \geq 1$) the series is **divergent**.	**2.** Answer to (a): since $\lvert -3/5\rvert$ is less than 1, the series is convergent, and its value equals $$\frac{18/25}{1-(-3/5)} = \frac{18/25}{8/5} = \frac{9}{20} \quad .$$ [**Added Dec. 3, 2019**: This corrects an error spotted by Professor Eli Rapaport, Ramapo College.] **Ans. to (b)**: Since $\lvert 7/6\rvert > 1$, the series is divergent (and there is no value of course).

Problem Type 11.2c: Determine whether the series is convergent or divergent if it is convergent, find its sum

$$\sum_{n=1}^{\infty} \frac{Aa^n + Bb^n}{c^n} \quad ,$$

(for some numbers a, b, c).

Example Problem 11.2c: Determine whether the series is convergent, and if it is, find its sum.

$$\sum_{n=1}^{\infty} \frac{4^n + 3(-6)^n}{7^n} \quad ,$$

Steps	**Example**
1. Split it up into two (or whatever) parts and treat each series on its own merit. If even one of them is divergent, then the whole thing is. If they are both (or all, in case there are more) convergent, then the whole thing is convergent.	**1.** $$\sum_{n=1}^{\infty} \frac{4^n + 3(-6)^n}{7^n} = \sum_{n=1}^{\infty} \frac{4^n}{7^n} + \sum_{n=1}^{\infty} 3\frac{(-6)^n}{7^n} = \quad ,$$ $$\sum_{n=1}^{\infty} (4/7)^n + \sum_{n=1}^{\infty} 3(-6/7)^n$$ Each of them is a geometrical series (the first one with $a = 4/7, r = 4/7$ the second one with $a = -18/7, r = -6/7$. Since $\lvert 4/7 \rvert < 1$ and $\lvert -6/7 \rvert < 1$, they are both convergent, so the whole thing is convergent.
2. If the whole thing is divergent (since at least one of them is) then you are done. Otherwise, find the sum of each series, and add them up, using the $a/(1-r)$ formula.	**2.** The total sum of the series is $$\frac{4/7}{1-4/7} + \frac{-18/7}{1-(-6/7)} = -2/39$$ **Ans.**: The series is convergent and its um is $-2/39$.