Dr. Z's Solutions to Dr. Scheffer's Math 152 Review Problems for Final Exam, Fall 2005

Corrected Dec. 16, 2012 [Thanks to Taylor Picillo, who corrected b_2 in(11b)]

FRs.tex

1. Let C be the curve $y = x^4/4$, with $0 \le x \le 1/2$.

(a) Set up an integral for the length of C.

(b) Using the binomial series and term-by-term integration, express the integral in part (a) as a convergent infinite series. Give numerical values for the first three terms in the series and a formula for the general term of the series.

(c) Explain why the method of (b) wouldn't work to find the length of the same curve extending from $x = 0$ all the way to $x = 2$. Give an approximate value for this length, using the trapezoidal rule with $n = 4$ divisions.

(d) Given that $\left| \frac{d^2}{dx^2} \sqrt{1+x^6} \right| \le 13$ for all $0 \le x \le 2$, estimate the error in your approximation in (c).

Sol. to 1

(a): Using the formula for the arclength

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \quad ,$$

with $f(x) = x^4/4$ and $a = 0$, $b = 1/2$, we get

Ans. to 1(a):

$$\int_0^{1/2} \sqrt{1+x^6}\, dx \quad ,$$

(b): Using the binomial series

$$(1+w)^k = \sum_{n=0}^{\infty} \binom{k}{n} w^n = 1 + kw + \frac{k(k-1)}{2!} w^2 + \ldots + \frac{k(k-1)\cdots(k-n+1)}{n!} w^n + \ldots \quad ,$$

(valid for $|w| < 1$), with $k = 1/2$ and $w = x^6$, we get

$$(1+x^6)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (x^6)^n = 1 + (1/2)x^6 + \frac{(1/2)(1/2-1)}{2!}(x^6)^2 +$$

$$\ldots + \frac{(1/2)(1/2-1)\cdots(3/2-n)}{n!}(x^6)^n + \ldots \quad ,$$

(valid for $|x| < 1$)

$$= 1 + (1/2)x^6 - (1/8)x^{12} + \ldots + \frac{(1/2)(1/2-1)\cdots(3/2-n)}{n!} x^{6n} + \ldots \quad ,$$

(valid for $|x| < 1$). Integrating **term by term** we get

$$\int_0^{1/2} \sqrt{1+x^6} =$$

$$\int_0^{1/2} \left(1 + (1/2)x^6 - (1/8)x^{12} + \ldots + \frac{(1/2)(1/2-1)\cdots(3/2-n)}{n!} x^{6n} + \ldots \right) dx$$

$$x+(1/2)\frac{x^7}{7}-(1/8)\frac{x^{13}}{13}+\ldots+\frac{(1/2)(1/2-1)\cdots(3/2-n)}{n!}\frac{x^{6n+1}}{6n+1}+\ldots|_0^{1/2}$$

$$=1/2+(1/2)\frac{(1/2)^7}{7}-(1/8)\frac{(1/2)^{13}}{13}+$$

$$\ldots+\frac{(1/2)(1/2-1)\cdots(3/2-n)}{n!}\frac{(1/2)^{6n+1}}{6n+1}+\ldots$$

$$=\frac{1}{2}+\frac{1}{2^8\cdot 7}-\frac{1}{2^{16}\cdot 13}+$$

$$\ldots+\frac{(1/2)(1/2-1)\cdots(3/2-n)}{n!}\frac{(1/2)^{6n+1}}{6n+1}+\ldots$$

This is the **Ans. to 1(b)**.

c): The radius of convergence of the Maclaurin series for $\sqrt{1+x^6}$ is 1, so it is only valid for the interval $-1<x<1$, and $x=2$ is outside it. The arclength is now

$$\int_0^2 \sqrt{1+x^6}\,dx \quad .$$

Using the Trapezoid rule with $n=4$, we get $\Delta x=(2-0)/4=1/2$ and the appx. is

$$\int_0^2 \sqrt{1+x^6} \approx \frac{1/2}{2}\left(\sqrt{1}+2\sqrt{1+(1/2)^6}+2\sqrt{1+(1)^6}+2\sqrt{1+(3/2)^6}+1\sqrt{1+(2)^6}\right)=$$

$$\frac{1}{4}\left(1+\frac{\sqrt{65}}{4}+2\sqrt{2}+\frac{\sqrt{793}}{4}+\sqrt{65}\right) \quad .$$

This is the **ans. to 1(c)**.

(d): Using

$$|E_T|\le\frac{K(b-a)^3}{12n^2} \quad ,$$

with $K=13$ (given by the problem), $n=4$, $a=0$ and $b=2$, we get

$$|E_T|\le\frac{13(2-0)^3}{12\cdot 4^2}=\frac{13}{24}$$

Ans. to 1(d): $|error|\le\frac{13}{24}$.

2. The curve with parametric equations

$$x = 8 - 2t^2, \quad y = \sin \pi t, \qquad -4 \le t \le 4$$

crosses itself at the origin. Find the t values at which it crosses the origin. Find the equations of both tangent lines at the origin.

Sol. to 2. Solving $x = 0$ gives $8 - 2t^2 = 0$ which gives $t = -2$ and $t = 2$. Now you have to check that y is also 0 for $t = -2$ and $t = 2$ (since we are talking about the origin). This is true. Now

$$\frac{dx}{dt} = -4t \quad , \frac{dy}{dt} = \pi \cos \pi t \quad ,$$

So the slope, dy/dx is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} == \frac{\pi \cos \pi t}{-4t} \quad .$$

Plugging-in $t = -2$ and $t = 2$ we get that the two slopes are $\frac{\pi \cos \pi 2}{-4 \cdot 2}$ and $\frac{\pi \cos \pi (-2)}{-4 \cdot (-2)}$ which are $-\pi/8$ and $\pi/8$. Using

$$(y - y_0) = m(x - x_0) \quad ,$$

for the equation of a line with slope m passing through a point (x_0, y_0), we get that the equations of the two tangents are

$$(y - 0) = (\pi/8)(x - 0) \quad , (y - 0) = (-\pi/8)(x - 0) \quad ,$$

Ans. to 2: $y = (\pi/8)x$ and $y = (-\pi/8)x$.

3. Find the solution of the differential equation $\frac{dy}{dx} = y\left(\frac{x^2-4x-9}{x^2-1}\right)$ with $y(2) = 3$. Give an explicit formula for y as a function of x. Graph the solution and determine the largest interval $A < x < B$ for which the solution exists.

Sol. to (3): Separating variables, we have

$$\frac{dy}{y} = \frac{x^2-4x-9}{x^2-1}\,dx$$

Integrating both sides we have

$$\int \frac{dy}{y} = \int \frac{x^2-4x-9}{x^2-1}\,dx$$

The integral on the left is easy: $\ln y$. The integral on the right requires first "long division" and then partial-fraction decomposition.

$$\frac{x^2-4x-9}{x^2-1} = 1 - \frac{4x+8}{x^2-1} \quad .$$

Now write:

$$\frac{4x+8}{x^2-1} = \frac{4x+8}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} \quad .$$

Equating numerators, we get:

$$4x+8 = A(x+1) + B(x-1) \quad .$$

Plugging-in "convenient values" $x = 1$ and $x = -1$ gives $A = 6$ and $B = -2$ so going back to the integrand on the right

$$\frac{x^2-4x-9}{x^2-1} = 1 - \frac{6}{x-1} + \frac{2}{x+1} \quad .$$

Integrating,

$$\int \frac{x^2-4x-9}{x^2-1}\,dx = x - 6\ln(x-1) + 2\ln(x+1) = x - \ln(x-1)^6 + \ln(x+1)^2 = x + \ln\frac{(x+1)^2}{(x-1)^6} + C$$

So we have

$$\ln y = x + \ln\frac{(x+1)^2}{(x-1)^6} + C \quad .$$

Exponentiating,

$$y = e^x \cdot e^{\ln\frac{(x+1)^2}{(x-1)^6} + C} = e^x \cdot e^{\ln\frac{(x+1)^2}{(x-1)^6}} \cdot e^C = Ce^x\frac{(x+1)^2}{(x-1)^6} \quad .$$

(Remember that $e^C = C$). Plugging-in $x = 2$ $y = 3$, we get

$$3 = Ce^2\frac{(2+1)^2}{(2-1)^6} = 9Ce^2 \quad .$$

Solving for C, we get: $C = 1/(3e^2)$. Going back we get the **solution**

$$y = e^x\frac{(x+1)^2}{3e^2(x-1)^6} \quad .$$

This function blows up at $x = 1$ so any interval avoiding $x = 1$ would do. Since the initial condition was $y(2) = 3$, our interval must contain $x = 2$ so the largest interval is $1 < x < \infty$.

4. Let R be the region in the *second* quadrant which is bounded by the curves $y = e^x$ and $y = 0$.

(a) Sketch the region R and find its area.

(b) Find the volume of the solids which result when the region R is revolved (1) about the x-axis; (2) about the y-axis. (Note that these integrals are improper.)

Sol. to (4a): You do the sketching! (remember that second quadrant means $x < 0$ and $y > 0$. The area is

$$\int_{-\infty}^{0} e^x\,dx = e^x|_{-\infty}^{0} = e^0 - e^{-\infty} = 1 - 0 = 1$$

Ans. to (4a): The area is 1.

Sol. to (4b)(1): The volume of the solid which results when the region R is rotated around the x-axis is

$$\pi \int_{-\infty}^{0} (e^x)^2\,dx = \pi \int_{-\infty}^{0} e^{2x}\,dx = \pi(\frac{e^{2x}}{2})|_{-\infty}^{0} = \pi(\frac{e^0}{2} - \frac{e^{-\infty}}{2}) = \pi(\frac{1}{2} - \frac{0}{2}) = \pi/2$$

Sol. to (4b)(2): The volume of the solid which results when the region R is rotated around the y-axis is

$$2\pi \int_{-\infty}^{0} (-x)(e^x)\,dx = -2\pi \int_{-\infty}^{0} xe^x\,dx$$

(Note that we have $-x$ instead of x since we are in the *second* quadrant where x is negaive.) So we have to evaluate the improper integral $\int_{-\infty}^{0} xe^x\,dx$. By integration by parts

$$\int xe^x\,dx = xe^x - e^x + C$$

So

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{R\to-\infty} \int_{R}^{0} xe^x\,dx = \lim_{R\to-\infty} (xe^x - e^x)|_{R}^{0} =$$

$$\lim_{R\to-\infty} [(0e^0 - e^0) - (Re^R - e^R)] = -1 + \lim_{R\to-\infty} (Re^R - e^R)$$

$$-1 + \lim_{R\to-\infty} \frac{R}{e^{-R}} - \lim_{R\to-\infty} e^R = -1 \quad ,$$

by L'Hôspital and 'pluging-in'. Combining we get

Ans. to (4b)(2): The volume of the solid which results when the region R is rotated around the y-axis is 2π.

5. Calculate the following indefinite integrals:

$$\text{(a)} \int \frac{e^x}{1+e^{2x}}\,dx \qquad \text{(b)} \int \sqrt{x}\sin(\sqrt{x})\,dx \qquad \text{(c)} \int \sqrt{5-4x-x^2}\,dx$$

(Suggestions: in (b), start by substituting $u=\sqrt{x}$; in (c), start by completing the square.)

Sol. to 5a). Do the substitution $u=e^x$, and det $du=e^x dx$ and so

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{du}{1+u^2} = \tan^{-1} u \quad ,$$

Going back to the x-language, we have

Ans. to 5a): $\tan^{-1}(e^x)+C$.

Sol. to 5b) Let's use t rather than u, since later on we need u for the integration-by-parts. With $t=x^{1/2}$ we get $dt/dx=(1/2)x^{-1/2}$ so $dx=2x^{1/2}dt=2tdt$ and the integral becomes

$$\int \sqrt{x}\sin(\sqrt{x})\,dx \;=\int 2t^2\sin t$$

This calls for **integration by parts** (twice!) First with $u=2t^2, v'=\sin t$, we get $u'=4t$ and $v=-\cos t$ so

$$\int 2t^2\sin t\,dt = 2t^2(-\cos t)+\int 4t\cos t\,dt = -2t^2\cos t+\int 4t\cos t\,dt \quad .$$

We need **integration-by-parts** one more time:

$$\int 4t\cos t\,dt \quad .$$

With $u=4t$ and $v'=\cos t$, we get $u'=4$ and $v=\sin t$, so

$$\int 4t\cos t\,dt = 4t\sin t-\int 4\sin t\,dt = 4t\sin t+4\cos t \quad .$$

Going back to the previous integral, we have

$$\int 2t^2\sin t\,dt = -2t^2\cos t+4t\sin t+4\cos t \quad .$$

and **now**, finally, we substitute $t=\sqrt{x}$ and get **Ans. to 5b)**:

$$-2x\cos\sqrt{x}+4\sqrt{x}\sin\sqrt{x}+4\cos\sqrt{x}+C \quad .$$

(Note: don't forget to add $+C$ at the very end).

Solution to (c).

$$5-4x-x^2 = -[x^2+4x-5] = -[(x+2)^2-4-5] = -[(x+2)^2-9] = 9-(x+2)^2 \quad .$$

So

$$\int \sqrt{5-4x-x^2}\,dx = \int \sqrt{9-(x+2)^2}\,dx \quad .$$

A natural **substitution** is $u = x+2$ with $du = dx$ so

$$\int \sqrt{5-4x-x^2}\,dx = \int \sqrt{9-u^2} \quad .$$

This calls for a **trig. subsitution**. $u = 3\sin\theta$. We have $du = 3\cos\theta d\theta$ and

$$\sqrt{9-u^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} = 3\cos\theta$$

We get

$$\int \sqrt{9-u^2}\,du = \int (3\cos\theta) 3\cos\theta d\theta = 9\int \cos^2\theta d\theta \quad .$$

Using the **trig. identity**

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad ,$$

we get

$$\frac{9}{2}\int (1+\cos(2\theta))\,d\theta = \frac{9}{2}(\theta + \frac{\sin(2\theta)}{2}) = \frac{9}{2}(\theta + \sin\theta\cos\theta)$$

(using yet another trig. identity: $\sin 2\theta = 2\sin\theta\cos\theta$) Going back to the u-language we get

$$\frac{9}{2}(\sin^{-1}(u/3) + (u/3)\sqrt{1-(u/3)^2}) = \frac{9}{2}\sin^{-1}(u/3) + \frac{u}{2}\sqrt{9-u^2}$$

Finally, going back to the x-language (recall that $u = x+2$)

Ans. to 5c):

$$\frac{9}{2}\sin^{-1}((x+2)/3) + \frac{x+2}{2}\sqrt{9-(x+2)^2} = \frac{9}{2}\sin^{-1}((x+2)/3) + \frac{x+2}{2}\sqrt{5-4x-x^2} + C \quad .$$

6. Find (a) $\displaystyle\int_0^{\pi/4} \tan^4 x\,dx$ and (b) $\displaystyle\int_0^{\pi/6} \sin^3 x \cos^4 x\,dx$.

Sol. of 6(a): We use the **trig. identity** $\tan^2 x = \sec^2 x - 1$ to write

$$\tan^4 x = \tan^2 x(\sec^2 - 1) = \tan^2 x \sec^2 x - \tan^2 x$$

$\tan^2 x$ still has an even power, so we use the above trig identity ($\tan^2 x = \sec^2 x - 1$) one more time to write

$$\tan^4 x = \tan^2 x(\sec^2 - 1) = \tan^2 x \sec^2 x - (\sec^2 x - 1) = \tan^2 x \sec^2 x - \sec^2 x + 1$$

For the first piece, we use the subsitution $u = \tan x$ giving $du = \sec^2 x dx$, so

$$\int \tan^2 x \sec^2 x\,dx = \int u^2 du = \frac{u^3}{3} = \frac{1}{3}\tan^3 x + C$$

$\int \sec^2 x\,dx$ is just $\tan x$ (for the formula sheet or from your memory) and $\int 1\,dx$ is x of course, so

$$\int \tan^4 x\,dx = \int \tan^2 x \sec^2 x\,dx - \int \sec^2 x\,dx + \int\,dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

Now inserting the limits of integration

$$\int_0^{\pi/4} \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x|_0^{\pi/4} =$$

$$(\frac{1}{3}\tan^3(\pi/4) - \tan(\pi/4) + \pi/4) - (\frac{1}{3}\tan^3 0 - \tan 0 + 0) = \frac{1}{3} - 1 + \pi/4 = \pi/4 - \frac{2}{3} \quad .$$

Ans. to 6(a): $\frac{\pi}{4} - \frac{2}{3}$.

Sol. to 6(b): Here sin is raised to an **odd** power, so we make the **substitution** $u = \cos x$. Now $du = -\sin x dx$ and so we write

$$\int \sin^3 x \cos^4 x\,dx = -\int \sin^2 x \cos^4 x(-\sin x dx) = -\int \sin^2 x u^4\,du \quad .$$

Using the famous **trig identity** $\sin^2 x = 1 - \cos^2 x$ we get that this equals

$$-\int (1-u^2)u^4\,du = \int (u^6 - u^4)\,du = \frac{u^7}{7} - \frac{u^5}{5} = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C \quad .$$

Now putting-in limits-of-integration

$$\int_0^{\pi/6} \sin^3 x \cos^4 x = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5}|_0^{\pi/6} = \frac{\cos^7(\pi/6)}{7} - \frac{\cos^5(\pi/6)}{5} - (\frac{\cos^7(0)}{7} - \frac{\cos^5(0)}{5})$$

$$= \frac{(\sqrt{3}/2)^7}{7} - \frac{(\sqrt{3}/2)^5}{5} - (\frac{\cos^7(0)}{7} - \frac{\cos^5(0)}{5})$$

$$= \frac{(\sqrt{3}/2)(3/4)^3}{7} - \frac{(\sqrt{3}/2)(3/4)^2}{5} - (\frac{1}{7} - \frac{1}{5}) = \frac{2}{35} - \frac{117\sqrt{3}}{4480}.$$

Ans. to 6b): $\frac{2}{35} - \frac{117\sqrt{3}}{4480}$.

7. Use geometric series to write the repeating decimal $2.171717\ldots$ as a fraction.

Sol. to 7):

$$2.171717\ldots = 2 + \frac{17}{100} + \frac{17}{10000} + \frac{17}{1000000} + \ldots =$$

$$= 2 + \frac{17}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \ldots\right)$$

Since $1/100 < 1$ we can use the famous geometric series sum

$$1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r} \quad ,$$

and we get

$$2.171717\ldots = 2 + \frac{17}{100} \cdot \frac{1}{1 - \frac{1}{100}} = 2 + \frac{17}{100} \cdot \frac{1}{\frac{99}{100}} = 2 + \frac{17}{99} = 2\frac{17}{99} = \frac{215}{99} \quad .$$

Ans. to 7): $\frac{215}{99}$.

8. A certain radio active substance is known to have half-life 1000 years and to decay at a rate which is always proportional to the amount present. If a sample contains 4 grams of the substance today, how much will be left in 500 years? How much was present in the sample 500 years ago? Give exact answers, not decimal approximations.

Sol. to 8.: Recall that the amount at time t, $M(t)$ is given by

$$M(t) = M_0 (1/2)^{t/HalfLife} \quad .$$

So in this problem:

$$M(t) = 4(1/2)^{t/1000} \quad .$$

In 500 years will be left

$$M(500) = 4(1/2)^{500/1000} = 4(1/2)^{1/2} = 4/\sqrt{2} = 2\sqrt{2}$$

500 years ago the amount was

$$M(-500) = 4(1/2)^{-500/1000} = 4\sqrt{2} \quad .$$

Ans. to 8.: $2\sqrt{2}$ and $4\sqrt{2}$ respectively.

9. (a) Does $\lim_{n\to\infty} \frac{\ln n}{n^2}$ exist? Explain your reasoning.

(b) Prove that $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges.

(c) Show that $\sum_{n=3}^{\infty} \frac{\ln n}{n^2} < \int_2^{\infty} \frac{\ln x}{x^2}\,dx$ by drawing areas related to the graph of $y = \frac{\ln x}{x^2}$.

Sol. to 9. a) Yes, since the corresponding limit with x replaced by n

$$\lim_{x\to\infty} \frac{\ln x}{x^2}$$

exists, by L'Hôpital and equals 0.

b) You could invoke Dr. Z's (p-q) test with $p = 2$ and $q = -1$, and since $p > 1$ it converges, but they want you to use the integral test. So let's consider the corresponding improper integral

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \int_2^{\infty} (\ln x)(x^{-2}) \quad .$$

First use integration by parts to do the indefinite integral. With $u = \ln x$ and $v' = x^{-2}$, we have $u' = 1/x$ and $v = -1/x$, so

$$\int (\ln x)(x^{-2}) = (\ln x)(-1/x) + \int x^{-2}\,dx = -\frac{\ln x}{x} - \frac{1}{x}$$

So

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{R\to\infty} \int_2^{R} \frac{\ln x}{x^2}\,dx = \lim_{R\to\infty} -\frac{\ln x}{x} - \frac{1}{x}\Big|_2^R$$

$$\lim_{R\to\infty} \left(-\frac{\ln R}{R} - \frac{1}{R}\right) + \frac{\ln 2}{2} + \frac{1}{2} = 0 + \frac{\ln 2}{2} + \frac{1}{2} \quad ,$$

By L'Hôpital and $\lim_{R\to\infty} 1/R = 0$. Since this is a **finite** number the integral converges, and by the integral test, the series converges.

10. Suppose you need numerical values of function $f(x)$ defined by a very complicated formula. You know, however, that $f(3) = 1$, $f'(3) = -2$ and $f''(3) = 20$. Moreover you know that the third derivative of $f(x)$ satisfies $|f'''(x)| \leq 24$ for all x in the interval $2 \leq x \leq 4$. Compute the second-degree Taylor polynomial T_2 for f centered at 3. Use it and Taylor's Inequality to solve the following problems.

(a) Calculate the best approximate value for $f(3.3)$ that you can from this information, and then estimate the error.

(b) Find a number $B > 0$ so that $|f(x) - T_2(x)| \leq 1/10$ for *all* numbers x in the interval $3 - B \leq x \leq 3 + B$.

Sol. to 10

$$T_2(x) = f(3)+f'(3)(x-3)+\frac{f''(3)}{2}(x-3)^2 = 1-2(x-3)+\frac{20}{2}(x-3)^2 = 1-2(x-3)+10(x-3)^2 \quad .$$

Plugging-in $M = 24$, $n = 2$ and $a = 3$ in

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad ,$$

we get

$$|R_3(x)| \leq \frac{24}{6}|x-3|^3 \quad ,$$

that simplifies to

$$|R_3(x)| \leq 4|x-3|^3 \quad ,$$

Sol. to 10a): Plugging in $x = 3.3$ we get

$$f(3.3) \approx 1 - 2(3.3-3) + 10(3.3-3)^2 = 1.3$$

$$|error| \leq |R_3(3.3)| \leq 4|3.3-3|^3 = \frac{27}{250} \quad .$$

Sol. to 10b): We need $4B^3 \leq 1/10$ so $B^3 \leq 1/40$ and we have
Ans. to 10b): $B = \frac{1}{\sqrt[3]{40}}$.

11. Let $f(x) = \cos(3x)$ and $g(x) = e^{x/2}$.

(a) Find the coefficients a_0, a_1, a_2 in the Maclaurin series $f(x)g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$.

(b) Find the coefficients b_0, b_1, b_2 in the Maclaurin series $\dfrac{f(x)}{g(x)} = b_0 + b_1 x + b_2 x^2 + \cdots$.

(You may obtain your answers either by algebraic manipulation of known power series or by the definition of the Maclaurin series.)

Sol. to 11): We only need to carry things up to x^2, and replace by $\ldots$ everything else.

$$f(x) = \cos(3x) = 1 - \frac{(3x)^2}{2} + \ldots \quad ; \quad g(x) = e^{x/2} = 1 + (x/2) + \frac{(x/2)^2}{2} + \ldots$$

Simplifying:

$$f(x) = 1 - \frac{9}{2}x^2 + \ldots \quad ; \quad g(x) = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \ldots$$

Multiplying:

$$f(x)g(x) = (1-\frac{9}{2}x^2+\ldots)(1+\frac{1}{2}x+\frac{1}{8}x^2) = (1+\frac{1}{2}x+\frac{1}{8}x^2)-\frac{9}{2}x^2(1+\ldots) = 1+\frac{1}{2}x-\frac{35}{8}x^2+\ldots$$

Ans. to (a): $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{-35}{8}$.

b): Division is much more complicated than multiplication, hence whenever you have a division, like here, try to make it multiplication. In this case it is possible: $f(x)/g(x) = \cos(3x)e^{-x/2}$, and things are very similar to the above.

$$f(x)/g(x) = (1-\frac{9}{2}x^2+\ldots)(1-\frac{1}{2}x+\frac{1}{8}x^2) = (1-\frac{1}{2}x+\frac{1}{8}x^2)-\frac{9}{2}x^2(1+\ldots) = 1-\frac{1}{2}x-\frac{45}{8}x^2+\ldots$$

Ans. to (b): $b_0 = 1$, $b_1 = -\frac{1}{2}$, $b_2 = \frac{-45}{8}$.

12. Use the formula for the sum of a geometric series to calculate the Maclaurin series for the function

$$f(x) = \frac{1}{3+2x^3}.$$

Write your answer in sigma notation. Use the result to find an infinite series representation for $\int_0^1 f(t)\,dt$. Estimate the size of the difference between this integral and the 3rd partial sum of the series.

Sol. to 12 Using

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$$

with $w = -2x^3/3$ we get

$$\frac{1}{3+2x^3} = \frac{1}{3(1+(2x^3/3))} = \frac{1}{3}\frac{1}{(1-(-2x^3/3))} = \frac{1}{3}\sum_{n=0}^{\infty}(\frac{-2x^3}{3})^n$$

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{3^{n+1}} x^{3n} \quad .$$

Integrating from 0 to 1, we get

$$\int_0^1 \frac{1}{3+2t^3} dt = \sum_{n=0}^{\infty} \frac{(-2)^n}{3^{n+1}} \int_0^1 t^{3n}\,dt \quad = \sum_{n=0}^{\infty} \frac{(-2)^n}{3^{n+1}} \frac{1}{3n+1} \quad = \sum_{n=0}^{\infty} \frac{(-2)^n}{(3n+1)3^{n+1}} \quad .$$

The first four terms, spelled-out are

$$\frac{(-2)^0}{(3\cdot 0+1)3^{0+1}} + \frac{(-2)^1}{(3\cdot 1+1)3^{1+1}} + \frac{(-2)^2}{(3\cdot 2+1)3^{2+1}} + \frac{(-2)^3}{(3\cdot 3+1)3^{3+1}}$$

$$= \frac{1}{3} + \frac{-1}{18} + \frac{4}{189} - \frac{4}{405}$$

The **difference** between the integral (which is the sum of the series) and the sum of the first **three** terms is the **next-in-line**, i.e. the **fourth** term, namely: $\frac{4}{405}$.

Ans. to second part of 12: the difference is $\leq 4/405$.

13. Determine if the following series are absolutely convergent, conditionally convergent, or divergent. In each case give details to support your answer and indicate which convergence test you are using.

(a) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n-1}\frac{n}{\ln n}$ (b) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n}\frac{n}{n^3+4}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{5^n\cdot(n!)^2}$

Sol to 13a): By L'Hôpital,

$$\lim_{n\to\infty}\frac{n}{\ln n}$$

equals ∞, hence

$$\lim_{n\to\infty}(-1)^{n-1}\frac{n}{\ln n}$$

does not exist, and the series **diverges by the divergence test**.

Ans. to 13a): series diverges by the divergence test.

Sol to 13b): First consider the **absolute version**

$$\sum_{n=2}^{\infty}\frac{n}{n^3+4} \quad .$$

By the **limit-comparison test** this has the same convergence status as

$$\sum_{n=2}^{\infty}\frac{n}{n^3}=\sum_{n=2}^{\infty}\frac{1}{n^2} \quad ,$$

which converges by the p-test ($p=2$). Since the absolute version is convergent, we have

Ans. to 13b): absolutely convergent.

Sol. to 13c): This calls for the **ratio test**.

$$a_n=\frac{(2n)!}{5^n(n!)^2}$$

Then

$$a_{n+1}=\frac{(2n+2)!}{5^{n+1}(n+1)!^2}$$

The ratio is

$$\frac{a_{n+1}}{a_n}=\frac{\frac{(2n+2)!}{5^{n+1}(n+1)!^2}}{\frac{(2n)!}{5^n(n!)^2}}$$

$$=\frac{(2n+2)!}{(2n)!}\cdot\frac{n!^2}{(n+1)!^2}\cdot\frac{5^n}{5^{n+1}}$$

$$=(2n+1)(2n+2)\cdot\frac{1}{(n+1)^2}\cdot\frac{1}{5}=\frac{(2n+1)(2n+2)}{5(n+1)^2}$$

This is the **simplified ratio**. Taking the limit, we get

$$\rho=\lim_{n\to\infty}\frac{(2n+1)(2n+2)}{5(n+1)^2}=\lim_{n\to\infty}\frac{(2n)(2n)}{5(n)^2}=\lim_{n\to\infty}\frac{4n^2}{5(n)^2}=\frac{4}{5} \quad .$$

Since $|\rho|<1$, we get

Ans. to 13c): Converges by the ratio test.

14. Use comparisons to determine whether the following improper integrals are convergent or divergent.

(a) $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x+3)}$ (b) $\displaystyle\int_0^\infty \frac{dx}{(4+x^2)^{3/2}}$

Sol. to 14a): By the limit-comparison test for integrals

$$\int_0^\infty \frac{dx}{(x+1)(x+3)}$$

has the same convergence status as

$$\int_1^\infty \frac{dx}{(x)(x)} = \int_1^\infty \frac{dx}{x^2} \quad ,$$

and this converges by the p-test for integrals ($p = 2$). Hence the original integral converges.

Sol. to 14b): By the limit-comparison test for integrals

$$\int_0^\infty \frac{dx}{(4+x^2)^{3/2}}$$

has the same convergence status as

$$\int_0^\infty \frac{dx}{(x^2)^{3/2}} = \int_0^\infty \frac{dx}{x^3}$$

and this converges by the p-test for integrals ($p = 3$). Hence the original integral converges.

15. Verify your answers to (a) and (b) in the preceding problem by calculating the integrals.

Sol. of 15a): By partial-fractions

$$\frac{1}{(x+1)(x+3)} = \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x+3}\right)$$

Integrating,

$$\int \frac{dx}{(x+1)(x+3)} = \frac{1}{2}\left(\int \frac{dx}{x+1} - \int \frac{dx}{x+3}\right) = \frac{1}{2}(\ln(x+1) - \ln(x+3)) = \frac{1}{2}\ln\left(\frac{x+1}{x+3}\right)$$

Now

$$\int_0^R \frac{dx}{(x+1)(x+3)} = \frac{1}{2}\ln\left(\frac{x+1}{x+3}\right)|_0^R = \frac{1}{2}\ln\left(\frac{R+1}{R+3}\right) - \frac{1}{2}\ln\left(\frac{0+1}{0+3}\right) -$$

$$= \frac{1}{2}\ln\left(\frac{R+1}{R+3}\right) + \frac{\ln 3}{2}$$

Finally

$$\int_0^\infty \frac{dx}{(x+1)(x+3)} = \frac{\ln 3}{2} + \lim_{R\to\infty} \frac{1}{2}\ln\left(\frac{R+1}{R+3}\right) =$$

$$\frac{\ln 3}{2} + \frac{1}{2}\ln\left(\frac{R}{R}\right) = \frac{\ln 3}{2} + 0 = \frac{\ln 3}{2} + 0 = \frac{\ln 3}{2} \quad .$$

Ans. to 15a): $(\ln 3)/2$.

Sol. to 15b) Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta d\theta$ and $4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta$, and $(4+x^2)^{3/2} = (4\sec^2\theta)^{3/2} = 8\sec^3\theta$. Also when $x = 0$, $\theta = 0$ and when $x = \infty$, $\theta = \pi/2$, so the integral becomes

$$\int_0^\infty \frac{dx}{(4+x^2)^{3/2}} = \int_0^{\pi/2} \frac{2\sec^2\theta}{8\sec^3\theta} = \int_0^{\pi/2} \frac{1}{4\sec\theta}$$

$$= \int_0^{\pi/2} \frac{\cos\theta}{4} = \frac{\sin\theta}{2}|_0^{\pi/2} = \frac{\sin(\pi/2)}{4} - \frac{\sin(0)}{4} = \frac{1}{4} \quad .$$

Ans. to 15b): $\frac{1}{4}$.

16. Show that $\sum_1^{\infty}\left(\sin^2\frac{1}{n}+\cos^2\frac{1}{n+1}-1\right)=\sin^2(1)$.

Sol. to 16): Somehow you have to make it a **telesoping series**. By the famous trig. identity $1-\cos^2 x=\sin^2 x$, we have

$$\sin^2\frac{1}{n}+\cos^2\frac{1}{n+1}-1=\sin^2\frac{1}{n}-(1-\cos^2\frac{1}{n+1})=\sin^2\frac{1}{n}-\sin^2\frac{1}{n+1}$$

It follows that

$$\sum_{n=1}^{\infty}\sin^2\frac{1}{n}+\cos^2\frac{1}{n+1}-1=\sum_{n=1}^{\infty}(\sin^2\frac{1}{n}-\sin^2\frac{1}{n+1})=$$

$$\sin^2 1-\sin^2(1/2)+\sin^2(1/2)-\sin^2(1/3)+\sin^2(1/3)-\sin^2(1/4)+\ldots$$

the partial sum is $\sin^2 1-\sin^2\frac{1}{n+1}$ and since the second term goes to 0, the sum of the series is the **first term of the first term**, namely $\sin^2 1$.

17. Determine the radius and interval of convergence of each of the following power series. In addition, determine those points at which each series is absolutely convergent.

(a) $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{\ln(n+2)}$ (b) $\sum_{n=1}^{\infty}\frac{(x+1)^n}{n^3 10^n}$

Sketch of the Sol. to 17a): By the ratio test, the limit of the ratios is $-x$, and setting its **absolute value** to be <1, we get $|x|<1$ and we get **radius of convergence** equals 1 and the **tentative interval of convergence** is $(-1,1)$. Plugging $x=-1$ gives you a (p,q) series with $p=0$ and $q=1$, and it follows by Dr. Z's p-q test that -1 is a point of divergence. When $x=1$ then the series does not converge absolutely (for the same reason) but by the **Alternating Series Test** it converges conditionally.

Ans. to 17a): Radius of convergence=1, interval of convergence $(-1,1]$. interval of absolute convergence $(-1,1)$.

Sketch of the Sol. to 17b): By the ratio test, the limit of the ratios is $(x+1)/10$, and setting its **absolute value** to be <1, we get $|x+1|<10$ and we get **radius of convergence** equals 10 and the **tentative interval of convergence** is $(-1-10,-1+10)=(-11,9)$. Plugging $x=-11$ gives you the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$$

which is **absolutely convergent** by the p-test ($p=3$), and hence, of course, convergent. Plugging $x=9$ gives you the series

$$\sum_{n=1}^{\infty}\frac{1}{n^3}$$

which is **absolutely convergent** by the p-test ($p=3$), and hence, of course, convergent (since right now the series is positive, it is the same thing).

Ans. to 17b): Radius of convergence=-1, interval of convergence $[-11,9]$. interval of absolute convergence $[-11,9]$.