Solutions to the"QUIZ" for Oct. 30, 2008

1. Sketch the curve $y = 2x^3 - 9x^2 + 12x - 2$.

Solution: $y' = 6x^2 - 18x + 12$, $y'' = 12x - 18$.

To get the **critical points** (numbers), we solve $y' = 0$, getting $6x^2 - 18x + 12 = 0$, which is the same as $6(x-1)(x-2) = 0$ yielding $x = 1$ and $x = 2$.

To get the (potential) **points of inflection** we set $y'' = 0$, getting $12x - 18 = 0$, which is $6(2x-3) = 0$ giving $x = 3/2$.

To find out whether $x = 1$ is (local) max or min, we plug into y'', getting $y''(1) = -6 < 0$ so it is a **local max**.

To find out whether $x = 2$ is (local) max or min, we plug into y'', getting $y''(2) = 6 > 0$ so it is a **local min**.

Now we find the corresponding y coordinates to these interesting numbers by plugging-in $y = 2x^3 - 9x^2 + 12x - 2$.

When $x = 1$, $y = 2 - 9 + 12 - 2 = 3$, So the point is $(1, 3)$ (local max)

When $x = 3/2$,

$$y = 2 \cdot (3/2)^3 - 9 \cdot (3/2)^2 + 12 \cdot (3/2) - 2 = (27/4) - (81/4) + 18 - 2 = -54/4 + 18 - 2 = -27/2 + 16 = 5/2$$

So the point is $(3/2, 5/2)$ (point of inflection)

When $x = 2$, $y = 2 \cdot 2^3 - 9 \cdot 2^2 + 12 \cdot 2 - 2 = 16 - 36 + 24 - 2 = 2$, so the point is $(2, 2)$ (local min).

To summarize:

Local max: $(1, 3)$. Local min: $(2, 2)$. Point of inflection: $(3/2, 5/2)$.

Now you draw it: from minus infinity it goes up to $(1, 3)$ (crossing the y axis at $(0, -2)$). From $(1, 3)$ it goes down to $(2, 2)$, passing through the point of inflection $(1.5, 2.5)$ (with slope $-3/2$), and from $(2, 2)$ it is increasing for ever after. It is concave down for $(-\infty, 3/2)$ and concave up for $(3/2, \infty)$.

(You draw it yourself!)

Comments: Most people approached it the right way, but only a half got it completely right, because of calculation errors. Please check your calculations, and use common sense to detect errors. For example, if there is only one max and one min, the min can't be bigger than the max.

2. Sketch the curve

$$y = \frac{1}{x^2 - 1} \quad .$$

Solution of 2: This is a **rational function**, and the most significant features are the **vertical asymptotes**, and after that, the **horizontal asymptotes**.

To get the vertical asymptotes, set the **denominator** equal to 0:

$$x^2 - 1 = 0$$

Solving, we get $(x-1)(x+1) = 0$, so $x = -1$ and $x = 1$.

To get the **horizontal asymptotes**, we take the limit $\lim_{x\to\infty}$ and $\lim_{x\to-\infty}$ (for rational functions, you always get the same answer).

$$\lim_{x\to\infty} \frac{1}{x^2 - 1} = \lim_{x\to\infty} \frac{1}{x^2} = \frac{1}{\infty^2} = \frac{1}{\infty} = 0 \quad .$$

(ditto for $\lim_{x\to-\infty} \frac{1}{x^2-1}$).

So far we have:

Vertical asymptotes: $x = -1$ and $x = 1$

Horizontal asymptotes: $y = 0$ (the x-axis), (both from the $-\infty$ and ∞ sides).

Even before differentiating, let's investigate how the function blows up at each vertical asymptote.

Immediately **before** $x = -1$, let's plug-in $x = -1.01$ (or whatever) and get that $y = 1/((-1.01)^2 - 1)$=VeryBigAndPositive, so **the curves goes up to heaven** when it reaches $x = -1$ from the left.

Immediately **after** $x = -1$, let's plug-in $x = -0.99$ (or whatever) and get that $y = 1/((-0.99)^2 - 1)$=VeryNegative, so **the curves comes up from hell** when it emerges from the blow-up at $x = -1$, heading right.

Immediately **before** $x = 1$, let's plug-in $x = 0.99$ (or whatever) and get that $y = 1/((0.99)^2 - 1)$=VeryNegative, so **the curves comes down to hell** when it reaches $x = 1$ from the left.

Immediately **after** $x = 1$, let's plug-in $x = 1.01$ (or whatever) and get that $y = 1/((1.01)^2 - 1)$=VeryPositive, so **the curves comes down from heaven** when it emerges from the blow-up at $x = 1$, heading right.

Finally, let't take the derivative (using either the chain rule or the quotient rule)

$$y' = \frac{-2x}{(x^2 - 1)^2} \quad .$$

Setting this equal to 0 (you only set the top equal to 0) gives $x = 0$, so this is a potential max or min. Plugging-it in into the function gives $y = -1$ so the only interesting point is $(0, -1)$. It is either max or min, which? In this case it makes more sense to use the first-derivative test (or better still, common sense). When $x = -0.01$ y' is positive, and when $x = 0.01$, y' is negative, so $x = 0$ is a transition from increasing to decreasing and hence a **local max**. This is also clear from common sense: On the open interval $-1 < x < 1$, the curve comes from hell at $x = -1$ and goes back to hell at $x = 1$, so it has to point down (i.e. be concave down).

To summarize:

Horiz. asymptotes: $y = 0$ (both on the left and right)

Vertical asymptotes:

$x = -1$ (goes up to heaven right before, emerges from hell right after).

$x = 1$ (goes down to hell right before, emerges from heaven right after).

Verbal desription:

Way back to the left the curve almost coincides with the negative x-axis (it is ever so slightly above), then right before $x = -1$ it climbs up to heaven, only to fall down to hell at $x = -1$, and works its way up from hell, until it reaches the **local max** $(0, -1)$, after which is starts to dive down to hell going to hell at $x = 1$, getting an upgrade to heaven at $x = 1$, and coming down from heaven immediately after $x = 1$ going down until it almost touches the x axes, but never actually making it there.

Now you draw it!

From the picture it is: concave up in the intervals $(-\infty, -1)$ and $(1, \infty)$, and concave down in the interval $(-1, 1)$.

Comments: Most people didn't have time to finish it, but quite a few people got it correctly, congratulations!, since I didn't give you enough time.