Solutions to the “QUIZ” for Oct. 20, 2008

1. Let \( f(x) = \sqrt{4 + x} \). Using the linear approximation of \( f(x) \) at \( a = 5 \) compute an approximation to \( f(4) \).

**First Solution of 1**: The **Linearization** of \( f(x) \) at \( x = a \), let’s call it \( L(x) \), is, in general

\[
L(x) = f(a) + f'(a)(x - a) .
\]

In this problem \( f(x) = (4 + x)^{1/2} \), and \( a = 5 \). We first need \( f'(x) \):

\[
f'(x) = (1/2)(4 + x)^{-1/2} = \frac{1}{2\sqrt{4 + x}} .
\]

plugging-in \( x = 5 \) we get:

\[
f'(5) = \frac{1}{2\sqrt{4 + 5}} = \frac{1}{2\sqrt{9}} = \frac{1}{6} .
\]

Also \( f(5) = \sqrt{9} = 3 \).

Plugging-in \( a = 5 \) and \( f'(5) = 1/6 \) into the general formula we have

\[
L(x) = 3 + \frac{1}{6}(x - 5) .
\]

This is the **linearization**. To get the **linear approximation** for \( f(4) \), we plug-in \( x = 4 \)

\[
L(4) = 3 + \frac{1}{6}(4 - 5) = 3 - \frac{1}{6} = \frac{17}{6} .
\]

Ans. to 1: \( \frac{17}{6} \).

**Second Solution of 1**: \[
\Delta f = f'(a)\Delta x .
\]

Here \( a = 5 \), \( \Delta x = 4 - 5 = -1 \). As before \( f'(x) = \frac{1}{2\sqrt{4 + x}} \), so \( f'(5) = \frac{1}{6} \), and

\[
\Delta f = \frac{1}{6} \cdot (-1) = -\frac{1}{6} .
\]

To get the **linear approximation** we add \( \Delta f \) to \( f(a) \), in this problem \( f(5) = \sqrt{9} = 3 \), so the linear approximation is \( f(a) + \Delta f \), which is \( 3 - \frac{1}{6} = \frac{17}{6} \).

Ans. to 1: \( \frac{17}{6} \).

**Comments**: About half of the students got it correctly. Some people only gave \( \Delta f \). This is not the **linear approximation**. The linear approximation is \( f(a) + \Delta f \). Many people got the sign wrong. They got \( \Delta f = 1/6 \) instead of \(-1/6\). Remember that \( \Delta x = 4 - 5 = -1 \) is **negative**.
2. Use Newton’s method with \( x_1 = 2 \) to find the second approximation to the equation \( x^4 - 17 = 0 \).

Solution to 2:

Newton’s law says

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} .
\]

Here we only need to do it once, since the first approx. \( x_1 \), is given by the problem. With \( n = 1 \) we have

\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} .
\]

Here \( f(x) = x^4 - 17 \), \( x_1 = 2 \). First we find \( f'(x) = 4x^3 \), and we have

\[
x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{2^4 - 17}{4 \cdot 2^3} =
\]

\[
2 - \frac{-1}{4 \cdot 2^3} = 2 - \frac{-1}{32} = \frac{65}{32} .
\]

Ans.: \( x_2 = \frac{65}{32} \).

Comments: Many people did it one more time. This was not asked for in this problem. If you had to also find \( x_3 \), then you would do it again with \( n = 2 \).