Solutions to the “QUIZ” for Nov. 10, 2008

1. Find $y = y(x)$ if $\frac{d^2y}{dx^2} = 6x$, $\frac{dy}{dx}(0) = 0$ and $y(0) = 0$.

Solution to 1: To get $\frac{dy}{dx}$, we take the anti-derivative of $6x$:

$$y'(x) = \int 6x \, dx = \frac{6x^2}{2} = 3x^2 + C.$$ 

To find $C$, we plug-in $x = 0$ and get, on the one hand:

$$y'(0) = 2 \cdot 0^2 + C = C$$

So $y'(0) = C$. On the other hand, from the problem: $y'(0) = 0$, so $C = 0$. Going back above, we get

$$y'(x) = 3x^2 + 0 = 3x^2.$$ 

To get $y(x)$, we take the anti-derivative of $y'(x)$, getting

$$y(x) = \int 3x^2 \, dx = \frac{3x^3}{3} = x^3 + C.$$ 

We still need to find the (new) $C$. Plugging-in $x = 0$ we get

$$y(0) = 0^3 + C = C.$$ 

On the other hand, by the data of the problem, $y(0) = 0$. So $C = 0$. Plugging-in above, we get:

$$y(x) = x^3 + 0 = x^3.$$ 

Ans. to 1: $y = x^3$.

2. Evaluate the following limit-Riemann sums by any method you like.

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left[ \frac{3i}{n} - \left( \frac{i}{n} \right)^2 \right] \frac{1}{n}.$$ 

Note: This material will be covered next time, as I finish section 5.1, so you weren’t yet ready to do it. For the sake of completeness, here is the solution.

First Solution of 2: You do pattern-recognition in the formal definition of the definite integral

$$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} h \sum_{i=1}^{n} f(a + ih)$$

where $h = \frac{b-a}{n}$. 
Let’s write the limit as
\[ \lim_{n \to \infty} \sum_{i=1}^{n} f(i/n) \frac{1}{n} . \]

Where \( f(x) = 3x - x^2 \). The sum goes from \( f(1/n) \), and since \( n \) is very big this is practically \( f(0) \) to \( f(1) \). So \( a = 0 \), \( b = 1 \) and \( h = 1/n \). So this complicated limit stands for the definite integral
\[ \int_{0}^{1} (3x - x^2) \, dx . \]

In section 5.3 we would learn how to compute it fast
\[ \int_{0}^{1} (3x - x^2) \, dx = 3 \frac{x^2}{2} - \frac{x^3}{3} \bigg|_{0}^{1} = 3 \cdot \frac{1}{2} - \frac{1}{3} - 0 = \frac{7}{6} . \]

Ans. to 2: \( \frac{7}{6} \).

**Second Solution of 2:** You do it directly. First let’s simplify the sum.
\[
\sum_{i=1}^{n} \left[ \frac{3i}{n} - \left( \frac{i}{n} \right)^2 \right]
\]
\[
= \sum_{i=1}^{n} \frac{3i}{n} - \sum_{i=1}^{n} \frac{i^2}{n^2}
\]
\[
= \frac{3}{n} \left( \sum_{i=1}^{n} i \right) - \frac{1}{n^2} \left( \sum_{i=1}^{n} i^2 \right)
\]

Using the standard formulas
\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2} ,
\]
\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} ,
\]

This equals
\[
= \frac{3}{n} \frac{n(n+1)}{2} - \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6}
\]

Going back to the limit, we have
\[
\lim_{n \to \infty} \frac{3}{n^2} \frac{n(n+1)}{2} - \lim_{n \to \infty} \frac{1}{n^2} \frac{(n+1)(2n+1)}{6}
\]
\[
\lim_{n \to \infty} \frac{3(n+1)}{2n} - \lim_{n \to \infty} \frac{(n+1)(2n+1)}{6n^2}
\]

By the **forget about the little ones**, this is:
\[
\lim_{n \to \infty} \frac{3n}{2n} - \lim_{n \to \infty} \frac{(n)(2n)}{6n^2} = \frac{3}{2} - \frac{1}{3} = \frac{7}{6} .
\]