Solutions to the "QUIZ" for Nov. 10, 2008

1. Find $y = y(x)$ if $\frac{d^2y}{dx^2} = 6x$, $\frac{dy}{dx}(0) = 0$ and $y(0) = 0$.

Solution to 1: To get $\frac{dy}{dx}$, we take the anti-derivative of $6x$:

$$y'(x) = \int 6x\,dx = \frac{6x^2}{2} = 3x^2 + C \quad .$$

To find C, we plug-in $x = 0$ and get, on the one hand:

$$y'(0) = 2 \cdot 0^2 + C = C$$

So $y'(0) = C$. On the other hand, from the problem: $y'(0) = 0$, so $C = 0$. Going back above, we get

$$y'(x) = 3x^2 + 0 = 3x^2 \quad .$$

To get $y(x)$, we take the anti-derivative of $y'(x)$, getting

$$y(x) = \int 3x^2\,dx = \frac{3x^3}{3} = x^3 + C \quad .$$

We still need to find the (new) C. Plugging-in $x = 0$ we get

$$y(0) = 0^3 + C = C \quad .$$

On the other hand, by the data of the problem, $y(0) = 0$. So $C = 0$. Plugging-in above, we get:

$$y(x) = x^3 + 0 = x^3 \quad .$$

Ans. to 1: $y = x^3$.

2. Evaluate the following limit-Riemann sums by any method you like.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{3i}{n} - \left(\frac{i}{n}\right)^2 \right] \frac{1}{n} \quad .$$

Note: This material will be covered next time, as I finish section 5.1, so you weren't yet ready to do it. For the sake of completeness, here is the solution.

First Solution of 2: You do *pattern-recognition* in the formal defition of the definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h \sum_{i=1}^{n} f(a + ih)$$

where $h = \frac{b-a}{n}$.

Let's write the limit as

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(i/n)\frac{1}{n} \quad .$$

Where $f(x) = 3x - x^2$. The sum goes from $f(1/n)$, and since n is very big this is practially $f(0)$ to $f(1)$. So $a = 0$, $b = 1$ and $h = 1/n$. So this complicated limit stands for the definite integral

$$\int_0^1 (3x - x^2)\, dx \quad .$$

In section 5.3 we would learn how to compute it fast

$$\int_0^1 (3x - x^2)\, dx = 3\frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = 3 \cdot \frac{1^2}{2} - \frac{1^3}{3} - 0 = \frac{7}{6} \quad .$$

Ans. to 2: $\frac{7}{6}$.

Second Solution of 2: You do it directly. First let's simplify the sum.

$$\sum_{i=1}^{n} \left[\frac{3i}{n} - \left(\frac{i}{n}\right)^2 \right]$$

$$= \sum_{i=1}^{n} \frac{3i}{n} - \sum_{i=1}^{n} \frac{i^2}{n^2}$$

$$= \frac{3}{n} \left(\sum_{i=1}^{n} i \right) - \frac{1}{n^2} \left(\sum_{i=1}^{n} i^2 \right)$$

Using the standard formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad ,$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad ,$$

This equals

$$= \frac{3}{n} \frac{n(n+1)}{2} - \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6}$$

Going back to the limit, we have

$$\lim_{n\to\infty} \frac{3}{n^2} \frac{n(n+1)}{2} - \lim_{n\to\infty} \frac{1}{n^2} \frac{(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty} \frac{3(n+1)}{2n} - \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

By the **forget about the little ones**, this is:

$$\lim_{n\to\infty} \frac{3n}{2n} - \lim_{n\to\infty} \frac{(n)(2n)}{6n^2} = \frac{3}{2} - \frac{1}{3} = \frac{7}{6} \quad .$$