Solutions to MATH 151(20-22), Dr. Z. , **Second Midterm**, Thursday, Nov. 20, 2008, and some suggested extra problems

Version of Fri., Nov. 28, 2008, 5:20pm (correcting minor typos in the extra problems for #7b, kindly pointed out by Andrew (Wei) Yang, and a few others I found myself)

Remember: 1. Show all your work. 2. Make sure that the answer(s) is (are) of the right type. If your answer will be of the wrong type (for example, if the answer is supposed to be an equation of a straight line, and your final answer is $y = x^2(x-2)+3$ (which is **not** an equation of straight line) you would get no points at all, even if everything is correct except for one step.

1. (16 points altogether) Consider the function

$$f(x) = \frac{x^2+1}{x^2-4}$$

(a) (2 points) Find the horizontal asymptotes

Sol. to 1a): Take the limit as $x \to \infty$

$$\lim_{x\to\infty} \frac{x^2+1}{x^2-4} = \lim_{x\to\infty} \frac{x^2}{x^2} = 1 \quad .$$

(Similary with $x \to -\infty$). So the Horizontal asympototes are $y = 1$ (on both sides).

(b) (2 points) Find the vertical asymptotes

Sol. of 2b): Set the **denominator** equal to 0:

$$x^2 - 4 = 0$$

and solve. Factoring, $(x-2)(x+2) = 0$, would get you that $x = -2$ and $x = 2$ are the Vertical Asymptotes.

(c) (2 points) Find all the local maxima and local minima

Sol. of 2c): We take the derivative, using the **quotinet rule**:

$$f'(x) = \left(\frac{x^2+1}{x^2-4}\right)' = \frac{(x^2+1)'(x^2-4)-(x^2+1)(x^2-4)'}{(x^2-4)^2} =$$

$$\frac{(2x)(x^2-4)-(x^2+1)(2x)}{(x^2-4)^2} = \frac{(2x)(x^2-4-x^2-1)}{(x^2-4)^2} = \frac{-10x}{(x^2-4)^2} \quad .$$

Setting this equal to 0, gives $x = 0$. To find whether it is a max or min it is easiest to use the first-derivative test (computing the second derivative is a big pain). When $x = -0.1$, $f'(x)$ is positive, and when $x = 0.1$, it is negative. So right before $x = 0$, the function is increasing, and

right after, it is decreasing. So $x = 0$ is a **local max**. To get the y-coordinate, we plug-in $x = 0$ into $f(x)$, getting $y = -\frac{1}{4}$. So the local max is $(0, -\frac{1}{4})$.

(d) (2 points) Find all the inflection points (if they exist)

Sol. to 1d: You could take the second derivative (using the quotient rule again, applied to $f'(x)$, but it is much more efficient to do the graph first, and notice that there are no inflection points from the graph.

Ans. to 1d: None.

(e) (4 points) In what intervals is the function (i) increasing? (ii) decreasing(?) (iii) concave up? (iv) concave down?

Sol. of 4e: The critical numbers are $x = -2$, $x = 0$, and $x = 2$. These divide the real line into **four** intervals: $(-\infty, -2)$,$(-2, 0)$,$(0, 2)$, and $(2, \infty)$.

For each of these we have to pick a random (easy) value and plug-in into

$$f'(x) = \frac{-10x}{(x^2 - 4)^2}$$

to see whether it is positive (meaning $f(x)$ is increasing), or negative (meaning $f(x)$ is decreasing).

$(-\infty, -2)$: pick $x = -10$, say, and get that $f'(x)$ is **positive**, so $f(x)$ is **increasing**.

$(-2, 0)$: pick $x = -1$ and get that $f'(x)$ is **positive**, so $f(x)$ is **increasing**.

$(0, 2)$: pick $x = 1$, and get that $f'(x)$ is **negative**, so $f(x)$ is **decreasing**.

$(2, \infty)$:pick $x = 10$, and get that $f'(x)$ is **negative**, so $f(x)$ is **decreasing**.

So

(i) increasing in the open intervals $(-\infty, -2)$,$(-2, 0)$,

(ii) decreasing in the open intervals $(0, 2)$, and $(2, \infty)$.

For (iii) and (iv), it is better to plot the graph first, and see from the graph.

(f) (4 points) Sketch the graph

The most important features are the vertical asymptototes. Plugging-in $x = -2.1$, we get that $f(x)$ is **positive** right before $x = -2$, which means that it shoots up to infinity right before the vertical asymptote $x = -2$. Plugging-in $x = -1.9$ into $f(x)$ gives that it is **negative** right after $x = -2$, meaning that it shoots up from $-\infty$ right after $x = -2$.

Plugging-in $x = 1.9$, we get that $f(x)$ is **negative** right before $x = 2$, which means that it shoots down to minus infinity right before the vertical asymptote $x = 2$. Plugging-in $x = 2.1$ into $f(x)$

gives that it is **positive** right after $x = 2$, meaning that it shoots down from ∞ right after $x = 2$. Combining with the vertical asymptotes at $y = 1$ from either sides, we have a graph with the following description (you plot it!)

At the far left it is almost along the horiz. line $y = 1$, then it starts to rise up-up making it to ∞ right at $x = 2$. At $x = 2$ itself it suddenly drops down to $-\infty$, and right after $x = 2$ it climbs its way up from $-\infty$ to the local max $(0, -1/4)$, where it starts descending again dropping to $-\infty$ at $x = 2$. At $x = 2$ it gets promoted to ∞, and right after $x = 2$ it starts dropping finally leveling off almost touching $y = 1$.

From the graph you can see that there are **no inflection points**, and that it is concave up in the open intervals $(-\infty, -2)$ and $(2, \infty)$ and concave down in $(-2, 2)$. Answering (d) and e(iii), e(iv).

Extra Practice Problems for the Final: Repeat exactly the same problem with the following rational functions:

$$f(x) = \frac{x}{x^2 - 1}$$
$$f(x) = \frac{x}{x^2 + 1}$$
$$f(x) = \frac{1}{x^3 - x}$$

2. (a) (6 points) Find

$$\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3}$$

Sol. of 6a: When $x = 0$ we get $0/0$, so we use L'Hôpital.

$$\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3} = \lim_{x \to 0} \frac{(x - \tan^{-1} x)'}{(x^3)'} = \lim_{x \to 0} \frac{1 - \frac{1}{1+x^2}}{3x^2} \quad .$$

We still get $0/0$, but **don't rush to use L'Hôpital again**. First **simplify!**, using algebra.

$$\lim_{x \to 0} \frac{\frac{x^2}{1+x^2}}{3x^2} = \lim_{x \to 0} \frac{1}{3(1 + x^2)}$$

Now **plug-in** $x = 0$, to get

$$= \frac{1}{3(1 + 0^2)} = \frac{1}{3} \quad .$$

Ans. to 2a: $\frac{1}{3}$.

(b) (6 points)

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6} - \frac{x^5}{120}}{x^7}$$

Sol. to 2b): Here we use L'Hôpital seven times!

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6} - \frac{x^5}{120}}{x^7}$$

$$= \lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}}{7x^6}$$

$$= \lim_{x\to 0} \frac{-\sin x + x - \frac{x^3}{6}}{7\cdot 6x^5}$$

$$= \lim_{x\to 0} \frac{-\cos x + 1 - \frac{x^2}{2}}{7\cdot 6\cdot 5x^4}$$

$$= \lim_{x\to 0} \frac{\sin x - x}{7\cdot 6\cdot 5\cdot 4x^3}$$

$$= \lim_{x\to 0} \frac{\cos x - 1}{7\cdot 6\cdot 5\cdot 4\cdot 3x^2}$$

$$= \lim_{x\to 0} \frac{-\sin x}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 3\cdot 2x}$$

$$= \lim_{x\to 0} \frac{-\cos x}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 3\cdot 2\cdot 1}$$

$$= \frac{-\cos 0}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 3\cdot 2\cdot 1} = -\frac{1}{5040}$$

Ans. to 2b): $-\frac{1}{5040}$.

Extra Practice for the Final: Compute

$$\lim_{x\to 0} \frac{x - \sin^{-1} x}{x}$$

$$\lim_{x\to 0} \frac{\sqrt{x+1} - \sqrt{x+2}}{x}$$

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$$

3. (a) (6 points) Use one step in Newton's method to approximate $\sqrt{15}$, starting with $x_0 = 4$.

Sol. to 3: Newton's method is a way to approximate roots of **equations** of the form $f(x) = 0$. So first, we must find an equation satisfied by $x = \sqrt{15}$. Squaring both sides, we get $x^2 = 15$, and moving 15 to the left, we get $x^2 - 15 = 0$. So $f(x) = x^2 - 15$ **not** (as many people took it) $f(x) = \sqrt{x}$.

Newton's method is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad .$$

With $n = 0$, it is:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad .$$

Since $f(x) = x^2 - 15$, we have that $f'(x) = 2x$. So

$$x_1 = x_0 - \frac{x_0^2 - 15}{2x_0} \quad .$$

Plugging-in $x_0 = 4$, we get

$$x_1 = 4 - \frac{4^2 - 15}{2 \cdot 4} = 4 - \frac{16 - 15}{2 \cdot 4} =$$
$$= 4 - \frac{1}{8} = \frac{31}{8} \quad .$$

Ans. to 3a: $\frac{31}{8}$.

Extra Problems for Practice for the Final

Repeat the same problem for:

$$\sqrt{26} \quad , x_0 = 5 \quad .$$
$$\frac{1}{\sqrt{50}} \quad , x_0 = 7 \quad .$$
$$9^{1/3} \quad x_0 = 2 \quad .$$

(b) (6 points) Use the linearization of $f(x) = \sqrt{x}$ at $x = 16$ to find an approximation to $\sqrt{15}$.

Sol. to 3b): Now $f(x) = \sqrt{x} = x^{1/2}$ and $a = 16$. Recalling the Linearization formula

$$L(x) = f(a) + f'(a)(x - a) \quad ,$$

we have $f'(x) = (1/2)x^{-1/2} = \frac{1}{2\sqrt{x}}$. So $f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$. Of course $f(16) = \sqrt{16} = 4$, so the **Linearization** is:

$$L(x) = 4 + \frac{1}{8}(x - 16) \quad .$$

To get an approximation for $\sqrt{15}$ we plug-in $x = 15$ (**Note: NOT** $x = \sqrt{15}$, this makes absolutely no sense! It is confusing input and output, the whole point is to find information about $\sqrt{15}$, so plugging-in is circular reasoning, logically, and does not make any sense mathematically).

So we get:

$$\sqrt{15} = f(15) \approx L(15) = 4 + \frac{1}{8}(15 - 16) = 4 - \frac{1}{8} = \frac{31}{8} \quad .$$

Ans. to 3b): $\frac{31}{8}$.

Extra Problems for Practice for the Final

Repeat the same problem with the following:

$f(x) = \ln x$ at $x = e^2$ to find an approximation to $\ln(e^2 + 0.1)$.

$f(x) = x^{1/3}$ at $x = 27$ to find an approximation to $26^{1/3}$.

4. (a) (6 points) The volume of a cube is expanding at a rate of 1 cubic centimeter per second. How fast is its surface area expanding when the volume of the cube is 1000 cubic centimeters?

Sol. to 4a: Let's the side of the cube be x. Its volume, V equals x^3, and its surface area, S equals $6x^2$. So, thanks to Geometry

$$V = x^3 \quad , \quad S = 6x^2 \quad .$$

Taking derivative with respect to t, we get

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} \quad , \quad \frac{dS}{dt} = 12x \frac{dx}{dt} \quad .$$

When $V = 1000$, $x = 10$. Plugging-in the first equation, using $\frac{dV}{dt} = 1$, we get

$$1 = 3 \cdot (10)^2 \frac{dx}{dt} \quad ,$$

So $\frac{dx}{dt} = \frac{1}{300}$. Plugging-into the second equation:

$$\frac{dS}{dt} = 12 \cdot 10 \cdot \frac{1}{300} = \frac{2}{5} \quad .$$

Ans. to 4a): $\frac{2}{5} cm^2/s$.

Comment: Another way of doing it is to use $V = x^3, S = 6x^2$ and algebra to express S in terms of V: $S = 6V^{2/3}$. Taking derivative with respect to t, we get:

$$\frac{dS}{dt} = 6(2/3)V^{-1/3} \frac{dV}{dt} = 4 \cdot (1000)^{-1/3} \cdot 1 = 4/10 = 2/5 \quad .$$

Extra Problems for Practice for the Final:

The volume of a sphere is expanding at a rate of π cubic centimeters per second. How fast is its surface area expanding when the volume of the sphere is $4000\pi/3$ cubic centimeters?

Hint: The volume of a sphere of radius r is $4\pi r^3/3$. Its surface area is $4\pi r^2$.

The surface area of a cube is changing at a rate of $2/5$ square centimeter per second. How fast is its volume expanding when the volume of the cube is 1000 cubic centimeters?

(b) (6 points) Find the linearization $L(x)$ of $f(x) = \ln x$ at $x = e^2$.

Sol. to 6b): Once again we use

$$L(x) = f(a) + f'(a)(x-a) \quad ,$$

Here $f(x) = \ln x$, $a = e^2$. $f(a) = \ln e^2 = 2$, $f'(x) = 1/x$, so $f'(e^2) = 1/e^2$. Combining, we have:

$$L(x) = \ln e^2 + \frac{1}{e^2}(x - e^2) \quad = 2 + \frac{1}{e^2}(x - e^2) \quad .$$

Ans. to 4b: $(L(x) = 2 + \frac{1}{e^2}(x - e^2)$.

Comment: Some people simplified further to $1 + x/e^2$, that's OK too, but not necessary, and the former form is better.

Extra Problems for Practice

Find the linearization $L(x)$ of $f(x) = e^x$ at $x = \ln 2$.

Find the linearization $L(x)$ of $f(x) = \tan x$ at $x = \pi/4$.

Find the linearization $L(x)$ of $f(x) = \tan^{-1} x$ at $x = 1$.

5. Differentiate the following functions

(a) (4 points)

$$f(x) = \ln x \cdot \tan^{-1}(x^2 + 1)$$

Sol. to 5a): First we use the **product rule**:

$$f'(x) = (\ln x \cdot \tan^{-1}(x^2+1))' = (\ln x)' \cdot \tan^{-1}(x^2+1) + (\ln x) \cdot (\tan^{-1}(x^2+1))'$$

$$\frac{1}{x} \cdot \tan^{-1}(x^2+1) + (\ln x) \cdot (\tan^{-1}(x^2+1))' \quad .$$

Next we use the **chain rule** applied to $(\tan^{-1}(x^2+1))'$.

$$= \frac{1}{x} \cdot \tan^{-1}(x^2+1) + (\ln x) \cdot \frac{1}{(x^2+1)^2+1} \cdot (2x)$$

Cleaning up, we get

$$f'(x) = \frac{\tan^{-1}(x^2+1)}{x} + \frac{2x \ln x}{(x^2+1)^2+1} \quad .$$

This is the **ans.** .

Extra Problems for Practice for the Final

Differentiate the following problems:

$$f(x) = e^x \cdot \sin^{-1}(x^2+1)$$

$$f(x) = \sin x^2 \cdot \sin^{-1}(x^2+1)$$

(b) (4 points) Differentiate wirh respect to x:

$$f(x) = 10^{\sin x} 5^{\cos x}$$

Sol. to 5b): The best way is to use **logarithmic differentiation**. Taking the ln of both sides, we get:

$$\ln f(x) = \ln 10^{\sin x} + \ln 5^{\cos x} = (\sin x)(\ln 10) + (\cos x)(\ln 5) = (\ln 10)\sin x + (\ln 5)(\cos x) \quad .$$

Differentiating both sides, we get

$$\frac{f'(x)}{f(x)} = (\ln 10)(\sin x)' + (\ln 5)(\cos x)' = (\ln 10)(\cos x) + (\ln 5)(-\sin x) = (\ln 10)\cos x - (\ln 5)\sin x \quad .$$

Finally, multiplying both sides by $f(x)$, we get

$$f'(x) = ((\ln 10)\cos x - (\ln 5)\sin x)10^{\sin x} 5^{\cos x} \quad .$$

Note: Another was of doing this is to use $10 = e^{\ln 10}$ and $5 = e^{\ln 5}$ to write

$$f(x) = e^{(\ln 10)\sin x} e^{(\ln 5)\cos x} = e^{(\ln 10)\sin x + (\ln 5)\cos x} \quad ,$$

and use the chain rule.

Extra Problem for Practice for the Final

Differentiate the following functions:

$$f(x) = 13^{\sin x} 8^{\tan x} 11^{e^x} \quad .$$

(c) (4 points) Differentiate with respect to x:

$$f(x) = (\sqrt{\ln(e^x)+1})^2$$

Sol. to 5c): Remember the slogan: "**Simplify before you differentiate!**" .

$$f(x) = (\sqrt{\ln(e^x)+1})^2 = \ln(e^x) + 1 = x + 1 \quad .$$

Now things are really easy: $f'(x) = 1$.

Ans. to 5c): $f'(x) = 1$.

Extra Problem for Practice for the Final

$$f(x) = (\sqrt{e^{\ln(\sin x)} + 1})^2 \quad .$$

6. (12 points) A jeweler has to design am open box, with a square base, of 2000 cubic centimeters, where the bottom is made of silver, and the four sides are made of gold. The cost of silver is 10 dollars per square centimeter, while the cost of gold is 20 dollars per square centimeter. What are the dimensions of the box that would **minimize** the cost?

Sol. to 6: The **constraint** is

$$x^2 y = 2000 \quad .$$

The area of the silver bottom is x^2, and its cost is $10x^2$. The area of each of the four sides (made of gold) is xy, so the total area of the sides is $4xy$, and their cost is $20(4xy) = 80xy$. The total cost is

$$C = 10x^2 + 80xy \quad .$$

From the constraint, we get $y = \frac{2000}{x^2}$. Pluging this into C we get the **goal function** purely in terms of x:

$$f(x) = 10x^2 + 80x \cdot \frac{2000}{x^2} = 10x^2 + 160000x^{-1} \quad .$$

To find the **minimum**, we find $f'(x)$

$$f'(x) = 20x - \frac{160000}{x^2}.$$

Setting this equal to 0, gives:

$$20x - \frac{160000}{x^2} = 0$$

Simplifying:

$$x^3 = 8000 \quad .$$

Taking the cubic root we get $x = 20$. Plugging into

$$y = \frac{2000}{x^2} \quad ,$$

we get

$$y = \frac{2000}{20^2} = \frac{2000}{400} = 5 \quad .$$

Ans. to 6: The dimensions that would minimize the cost are $20 \times 20 \times 5$ cm.

Extra Problems for Practice for the Final

Repeat the same problem for a closed box, where both top and bottom are made of silver.

Repeat the original porblem, but where the cost is fixed at 12000 dollars and you have to maximize the volume.

7. (a) (6 points) Find the absolute maximum and the absolute minimum of the function

$$f(x) = xe^{-2x} \quad ,$$

on the interval $[0, 2]$.

Sol. of 7a): First we take the derivative, using the product rule (and the chain rule)

$$f'(x) = e^{-2x} + xe^{-2x}(-2) = e^{-2x} - 2xe^{-2x} \quad .$$

Now we **simplify as much as we can**:

$$f'(x) = e^{-2x}(1 - 2x)$$

Setting $f'(x) = 0$, we get $1 - 2x = 0$, so $x = \frac{1}{2}$ is a critical point. Since it happens to be in our interval, $[0, 2]$, we take it as finalist. (Note that e^{-2x} is never zero). The **finalists** are the critical point(s) (that happen to belong to our interval) and the **two endpoints**.

The final contestnats are $x = 0, x = \frac{1}{2}, x = 2$. Plugging-in $f(x)$ we get

$$f(0) = 0 \quad , \quad f(\frac{1}{2}) = \frac{1}{2e} \quad , \quad f(2) = \frac{2}{e^4} \quad .$$

The **abs.** min. is obviously 0 at $x = 0$. The abs. max is $\frac{1}{2e}$ at $x = \frac{1}{2}$. (Note, here you use that $e = 2.71...$)

Extra Problems for Practice for the Final

Repeat the above problem with:

$f(x) = x^2e^{-3x}$, $[0, 3]$.

$f(x) = xe^{-x^2}$, $[-1, 1]$.

(b) (6 points) Solve the differntial equation

$$y''(x) = \sin x \quad ,$$

with initial condition $y(\pi/2) = 1$, $y'(\pi/2) = 2$.

Sol. of 6b): First we must find $y'(x)$, by finding the anti-derivative of $y''(x)$:

$$y'(x) = \int \sin x \, dx = -\cos x + C \quad .$$

Plugging-in $x = \pi/2$ and using the data that $y'(\pi/2) = 2$, we get

$$2 = -\cos(\pi/2) + C = 0 + C = C \quad .$$

So $C = 2$. Now we know for sure that

$$y'(x) = -\cos x + 2 \quad .$$

To find $y(x)$, we take the anti-derivative of that:

$$y(x) = \int (-\cos x + 2)\, dx = -\sin x + 2x + C$$

To find C we plug-in $x = \pi/2$, using $y(\pi/2) = 1$:

$$1 = -\sin(\pi/2) + 2 \cdot (\pi/2) + C$$

So

$$1 = -1 + \pi + C$$

Solving for C gives:

$$C = 2 - \pi \quad .$$

Putting it back in the formula for $y(x)$ above, we get

$$y(x) = -\sin x + 2x + 2 - \pi \quad .$$

Ans. to 7b): $y(x) = -\sin x + 2x + 2 - \pi$.

Extra Problems for Practice for the Final

Solve the differential equations

$$y'' = e^x \quad , y(0) = 1, y'(0) = e + 1 \quad .$$

$$y' = \sec^2 x + 1, y(\pi/4) = 1 + \pi/4, \quad .$$

8. (12 points) Use the definition of the definite integral

$$\int_a^b f(x)dx \; = \; \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad ,$$

(where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$), to evaluate the definite integral

$$\int_0^2 3x dx \quad .$$

Hint: You may need the formula:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad .$$

Sol. of 8.: Here $a = 0, b = 2, f(x) = 3x$. $\Delta x = (2-0)/n$.

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^{n} f(0 + \frac{2}{n}i)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(\frac{2}{n}i)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 3(\frac{2}{n}i)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 3(\frac{2}{n})\frac{2}{n}i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{12}{n^2}i \quad .
\end{aligned}$$

Everything not involving i can be taken in front of the $\sum$:

$$= \lim_{n\to\infty} \frac{12}{n^2}\left(\sum_{i=1}^{n} i\right) \quad .$$

Using the formula, this equals

$$= \lim_{n\to\infty} \frac{12}{n^2}\frac{n(n+1)}{2}$$

Simplifying:

$$= \lim_{n\to\infty} \frac{6(n+1)}{n} \quad .$$

Using "forget about the little ones", this equals

$$= \lim_{n\to\infty} \frac{6n}{n} = \lim_{n\to\infty} 6 = 6 \quad .$$

Ans. to 8.: 6.

Extra Problems for Practice for the Final:

Repeat the same problem for the following definite integrals

$$\int_0^3 x^2\, dx$$

$$\int_{-1}^1 (3-x)\, dx$$

$$\int_0^2 x^3\, dx$$