1. (12 points) Find $f'(1)$, if  

$$f(x) = \frac{1}{x^2}$$

from the definition of the derivative [No Credit for other methods].

Sol. to 1):

$$f'(1) = \lim_{h \to 0} \frac{\frac{1}{(1+h)^2} - \frac{1}{1}}{h} = \lim_{h \to 0} \frac{1 - (1+h)^2}{h(1+h)^2} = \lim_{h \to 0} \frac{1 - (1+2h+h^2)}{h(1+h)^2}$$

$$= \lim_{h \to 0} \frac{1 - 1 - 2h - h^2}{h(1+h)^2} = \lim_{h \to 0} \frac{-2h - h^2}{h(1+h)^2}$$

$$= \lim_{h \to 0} \frac{-h(2+h)}{h(1+h)^2} = \lim_{h \to 0} \frac{-2(1+h) + 2}{(1+h)^2} = \lim_{h \to 0} \frac{-2 + 2}{1+h} = -2$$

Ans. to 1: $-2$.

Comment: Most people first found $f'(x)$ and only at the end plugged in $x = 1$. This is correct, but it is less efficient.

2. (12 points) Find $\frac{dy}{dx}$ by implicit differentiation if

$$x^2y + 2xy + xy^3 = 5$$

Sol. to 2:

$$(x^2y + 2xy + xy^3)' = 5'$$

yields

$$(x^2y)' + (2xy)' + (xy^3)' = 0$$

By the product rule

$$(x^2)'y + x^2y' + (2x)'y + 2xy' + x'y^3 + x(y^3)' = 0$$

Using both straightforward differentiations and the chain-rule, we have:

$$2xy + x^2y' + 2y + 2xy' + y^3 + x3y^2y' = 0$$

So much for calculus. Now we have to solve for $y'$. Keep all the terms involving $y'$ at the left, and move everything else to the right:

$$x^2y' + 2xy' + 3xy^2y' = -(2xy + 2y + y^3)$$
Factoring-out $y'$ on the left we get

$$(x^2 + 2x + 3xy^2)y' = -(2xy + 2y + y^3)$$

and finally, dividing both sides by the factor in front of $y'$ gives

$$y' = -\frac{2xy + 2y + y^3}{x^2 + 2x + 3xy^2}.$$  

Ans. to 2): $\frac{dy}{dx} = -\frac{2xy + 2y + y^3}{x^2 + 2x + 3xy^2}$.

3. (12 points [4 points each]) If the Law of Motion of a particle is $s(t) = t^3 - 3t^2 + 3t$, find

(a) The speed and direction (forward or backwards) at $t = 2$.

Sol. to 3a): $v(t) = 3t^2 - 6t + 3 = 3(t - 1)^2$, so $v(2) = 3 \cdot (2 - 1)^2 = 3$.

Ans. to 3a: The particle is moving forward at a speed of 3.

(b) The time(s) when it is at rest.

Sol. to 3b): Solving $v = 0$, we get $3(t - 1)^2 =$ whose solution is $t = 1$.

Ans. to 3b: The particle is at rest when $t = 1$.

(c) The total distance travelled between $t = -1$ and $t = 1$.

Sol. to 3c): Since the velocity is always positive (except at $t = 1$ when it is at rest), the particle is moving forward all the time, so one does need to break-up the journey from $t = -1$ to $t = 1$ (like we would normally have to), into forward epochs and backwards one. The answer is simply $s(1) - s(-1) = 1 - (-7) = 8$.

Ans. to 3c): The distance travelled from $t = -1$ to $t = 1$ was 8.

4. (15 points ([5 pts each]) Find the derivative $f'(x)$ if:

(a) $f(x) = \frac{\cos x}{2x + 1}$

Sol. of 4a):

$$f'(x) = \frac{(\cos x)'(2x + 1) - (\cos x)(2x + 1)'}{(2x + 1)^2} = \frac{(-\sin x)(2x + 1) - (\cos x) \cdot 2}{(2x + 1)^2}$$

$$= \frac{-(2x + 1) \sin x - 2 \cos x}{(2x + 1)^2} = -\frac{(2x + 1) \sin x + 2 \cos x}{(2x + 1)^2}$$
Ans. to 4a): $f'(x) = -\frac{(2x+1)\sin x + 2 \cos x}{(2x+1)^2}$.

(b) $f(x) = x \sin x \cos x$

Sol. of 4b): First Way: We have to use the product rule twice.

$$f'(x) = (x \sin x)' \cos x + (x \sin x)(\cos x)' = (x' \sin x + x (\sin x)') \cos x + (x \sin x)(- \sin x) =$$

$$(\sin x + x \cos x) \cos x - x \sin^2 x = \sin x \cos x + x \cos^2 x - x \sin^2 x$$

Ans. to 4b): $f'(x) = \sin x \cos x + x \cos^2 x - x \sin^2 x$.

Second Way: Use the trig. identity $\sin 2x = 2 \sin x \cos x$ to write $\sin x \cos x = (1/2) \sin 2x$, so

$$f(x) = (x/2) \sin 2x.$$ Now use the product rule (and the chain rule for $(\sin 2x)' = 2 \cos 2x$.

(c) $f(x) = \frac{e^x}{1+3x}$

Sol. to 5c): By the quotient rule:

$$f'(x) = \frac{(e^x)'(1 + 3x) - (e^x)(1 + 3x)'}{(1 + 3x)^2} = \frac{e^x(1 + 3x) - (e^x)3}{(1 + 3x)^2} =$$

$$\frac{e^x(3x - 2)}{(1 + 3x)^2}$$

Ans. to 5c): $f'(x) = \frac{e^x(3x - 2)}{(1 + 3x)^2}$.

5. (13 points) Find an equation of the tangent line to the curve

$$y = x^3 + 2x + 1,$$

at the point (1, 4).

Solution to 5: The answer should be an equation of a straight line.

First let’s take the derivative:

$$\frac{dy}{dx} = 3x^2 + 2.$$

At the point (1, 4), we have $x = 1$, so to get the slope at that point, we plug-in $x = 1$:

$$m = \frac{dy}{dx} \bigg|_{x=1} = 3 \cdot 1^2 + 2 = 3 + 2 = 5.$$

Using the point-slope equation for a straight line, we have:

$$(y - 4) = 5(x - 1),$$

3
and simplifying, we get
\[ y - 4 = 5x - 5 , \]
and finally,
\[ y = 5x - 1 . \]

Ans. to 5: \( y = 5x - 1 \).

6. (12 points [3 pts each]) Find the limits

(a) \( \lim_{x \to -2} \frac{x^2 - 1}{x^2 + 2x + 3} \)

Sol. to 6a):
\[
\lim_{x \to -2} \frac{x^2 - 1}{x^2 + 2x + 3} = \frac{(-2)^2 - 1}{(-2)^2 + 2 \cdot (-2) + 3} = \frac{3}{3} = 1 .
\]

Ans. to 6a): 1.

(b) \( \lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x} \)

Sol. to 6b: Multiplying by the conjugate:
\[
\lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x} = \lim_{x \to 0} \frac{(\sqrt{4 + x} - 2)(\sqrt{4 + x} + 2)}{x(\sqrt{4 + x} + 2)}
\]
\[
= \lim_{x \to 0} \frac{(\sqrt{4 + x})^2 - 2^2}{x(\sqrt{4 + x} + 2)} = \lim_{x \to 0} \frac{(4 + x) - 4}{x(\sqrt{4 + x} + 2)}
\]
\[
= \lim_{x \to 0} \frac{1}{x(\sqrt{4 + x} + 2)} = \lim_{x \to 0} \frac{1}{\sqrt{4 + 0} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2 + 2} = \frac{1}{4} .
\]

Ans. to 6b: \( \frac{1}{4} \).

(c) \( \lim_{x \to \pi/2} \frac{1 - \cos x}{x} \)

Sol. to 6c:
\[
\lim_{x \to \pi/2} \frac{1 - \cos x}{x} = \frac{1 - \cos \pi/2}{\pi/2}
\]
\[
= \frac{1 - 0}{\pi/2} = \frac{2}{\pi} .
\]

Ans. to 6c: \( \frac{2}{\pi} \).
(d) \[
\lim_{x \to 0} \frac{\sin 10x}{\sin 5x}
\]

Sol. to 6d: By Dr. Z’s get-rid of sines rule:

\[
\lim_{x \to 0} \frac{\sin 10x}{\sin 5x} = \lim_{x \to 0} \frac{10x}{5x} = \lim_{x \to 0} 2 = 2.
\]

Ans.to 6d: 2.

7. (12 points) Let

\[
f(x) = \begin{cases} 
\sqrt{-2x}, & \text{if } x < 0; \\
3 + 2x, & \text{if } 0 \leq x < 3; \\
(2x - 3)^2, & \text{if } x \geq 3.
\end{cases}
\]

(a) [1 pt. each] For each of the following limits, evaluate it, if it exists.

(i) \[
\lim_{x \to 0^+} f(x)
\]

Ans. to (i): \(3 + 2 \cdot 0 = 3\).

(ii) \[
\lim_{x \to 0^-} f(x)
\]

Ans. to (ii): \(\sqrt{-2} \cdot 0 = \sqrt{0} = 0\).

(iii) \[
\lim_{x \to 0} f(x)
\]

Ans. to (iii): Does Not Exist (since 0 and 3 are different).

(iv) \[
\lim_{x \to 3^+} f(x)
\]

Ans. to (iv): \((2 \cdot 3 - 3)^2 = 9\).

(v) \[
\lim_{x \to 3^-} f(x)
\]

Ans. to (v): \(3 + 2 \cdot 3 = 9\).

(vi) \[
\lim_{x \to 3} f(x)
\]

Ans. to (vi): 9 (the common value of (iv) and (v)).
(b) [6 pts] Where is $f$ discontinuous? Explain! (no credit without explanation).

**Solution to 7b):** The only place where $f(x)$ is discontinuous is at $x = 0$, since there the limit of the function as $x$ goes to 0 does not exist. In all other places the limit exists and equals to the function-value, since each piece is continuous, and the interface at $x = 3$ is also ok, since $\lim_{x \to 3}$ exists and equals $f(3)$ (both being 9).

8. (12 points) Find the point on the curve

$$y = x^2 - 3x + 2$$

where the tangent line is parallel to the line $y = x + 4$. Then find the equation of that tangent line.

**Solution to 8:** The slope of the line $y = x + 4$ is 1, so we have to set the derivative $\frac{dy}{dx}$ equal to 1.

First, let’s find the derivative:

$$\frac{dy}{dx} = 2x - 3$$

Setting this equal to 1

$$2x - 3 = 1$$

and solving for $x$ gives

$$2x = 4$$

and, so

$$x = 2$$

To get the $y$-coordinate, we plug into $y$

$$\left.\left(x^2 - 3x + 2\right)\right|_{x=2} = 2^2 - 3 \cdot 2 + 2 = 4 - 6 + 2 = 0$$

So the point is $(2,0)$.

**Ans. to First Part of 8:** The point is $(2,0)$.

We still need to find the equation of the tangent line. Using

$$(y - y_0) = m(x - x_0)$$

we get

$$(y - 0) = 1 \cdot (x - 2)$$

that yields:

$$y = x - 2$$

**Ans. to Second Part of 8:** The equation of the tangent line to the curve $y = x^2 - 3x + 2$ parallel to $y = x + 4$ is $y = x - 2$. 