MATH 151(20-22), Dr. Z. , **Solutions to First Midterm**, Mon., Oct. 13, 2008.

1. (12 points) Find $f'(1)$, if

$$f(x) = \frac{1}{x^2}$$

from the definition of the derivative [No Credit for other methods].

Sol. to 1):

$$f'(1) = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - \frac{1}{1^2}}{h} = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - 1}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{1-(1+h)^2}{(1+h)^2}}{h} == \lim_{h\to 0} \frac{1-(1+h)^2}{h(1+h)^2} = \lim_{h\to 0} \frac{1-(1+2h+h^2)}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{1-1-2h-h^2}{h(1+h)^2} = \lim_{h\to 0} \frac{-2h-h^2}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{-h(2+h)}{h(1+h)^2} = \lim_{h\to 0} \frac{-(2+h)}{(1+h)^2} = \lim_{h\to 0} \frac{-(2+0)}{(1+0)^2} = -2 \quad .$$

Ans. to 1: -2.

Comment: Most people first found $f'(x)$ and only at the end plugged in $x = 1$. This is correct, but it is less efficient.

2. (12 points) Find $\frac{dy}{dx}$ by implicit differentiation if

$$x^2y + 2xy + xy^3 = 5 \quad .$$

Sol. to 2:

$$(x^2y + 2xy + xy^3)' = 5' \quad ,$$

yields

$$(x^2y)' + (2xy)' + (xy^3)' = 0 \quad ,$$

By the product rule

$$(x^2)'y + x^2y' + (2x)'y + 2xy' + x'y^3 + x(y^3)' = 0 \quad .$$

Using both straightforward differentiations and the chain-rule, we have:

$$2xy + x^2y' + 2y + 2xy' + y^3 + x3y^2y' = 0 \quad .$$

So much for **calculus**. Now we have to **solve for** y'. Keep all the terms involving y' at the left, and move everything else to the right:

$$x^2y' + 2xy' + 3xy^2y' = -(2xy + 2y + y^3) \quad .$$

Factoring-out y' on the left we get

$$(x^2 + 2x + 3xy^2)y' = -(2xy + 2y + y^3) \quad ,$$

and finally, dividing both sides by the factor in front of y' gives

$$y' = -\frac{2xy + 2y + y^3}{x^2 + 2x + 3xy^2} \quad .$$

Ans. to 2): $\frac{dy}{dx} = -\frac{2xy+2y+y^3}{x^2+2x+3xy^2}$.

3. (12 points [4 points each]) If the Law of Motion of a particle is $s(t) = t^3 - 3t^2 + 3t$, find

(a) The speed and direction (forward or backwards) at $t = 2$.

Sol. to 3a): $v(t) = 3t^2 - 6t + 3 = 3(t-1)^2$, so $v(2) = 3 \cdot (2-1)^2 = 3$.

Ans. to 3a: The particle is moving forward at a speed of 3.

(b) The time(s) when it is at rest.

Sol. to 3b): Solving $v = 0$, we get $3(t-1)^2 =$ whose solution is $t = 1$.

Ans. to 3b: The particle is at rest when $t = 1$.

(c) The total distance travelled between $t = -1$ and $t = 1$.

Sol. to 3c): Since the velocity is always positive (except at $t = 1$ when it is at rest), the particle is moving forward all the time, so one does need to break-up the journey from $t = -1$ to $t = 1$ (like we would normally have to), into forward epochs and backwards one. The answer is simply $s(1) - s(-1) = 1 - (-7) = 8$.

Ans. to 3c): The distance travelled from $t = -1$ to$t = 1$ was 8.

4. (15 points ([5 pts each]) Find the derivative $f'(x)$ if:

(a) $f(x) = \frac{\cos x}{2x+1}$

Sol. of 4a):

$$f'(x) = \frac{(\cos x)'(2x+1) - (\cos x)(2x+1)'}{(2x+1)^2} = \frac{(-\sin x)(2x+1) - (\cos x) \cdot 2}{(2x+1)^2}$$

$$= \frac{-(2x+1)\sin x - 2\cos x}{(2x+1)^2} = -\frac{(2x+1)\sin x + 2\cos x}{(2x+1)^2}$$

Ans. to 4a): $f'(x) = -\frac{(2x+1)\sin x + 2\cos x}{(2x+1)^2}$.

(b) $f(x) = x \sin x \cos x$

Sol. of 4b): First Way: We have to use the product rule **twice**.

$$f'(x) = (x \sin x)' \cos x + (x \sin x)(\cos x)' = (x' \sin x + x(\sin x)') \cos x + (x \sin x)(-\sin x) =$$

$$(\sin x + x \cos x) \cos x - x \sin^2 x = \sin x \cos x + x \cos^2 x - x \sin^2 x$$

Ans. to 4b): $f'(x) = \sin x \cos x + x(\cos^2 x - \sin^2 x)$.

Second Way: Use the trig. identity $\sin 2x = 2 \sin x \cos x$ to write $\sin x \cos x = (1/2) \sin 2x$, so $f(x) = (x/2) \sin 2x$. Now use the product rule (and the chain rule for $(\sin 2x)' = 2 \cos 2x$.

(c) $f(x) = \frac{e^x}{1+3x}$

Sol. to 5c):By the quotient rule:

$$f'(x) = \frac{(e^x)'(1+3x) - (e^x)(1+3x)'}{(1+3x)^2} = \frac{e^x(1+3x) - (e^x)3}{(1+3x)^2} =$$

$$\frac{e^x(3x-2)}{(1+3x)^2}$$

Ans. to 5c): $f'(x) = \frac{e^x(3x-2)}{(1+3x)^2}$.

5. (13 points) Find an equation of the tangent line to the curve

$$y = x^3 + 2x + 1 \quad ,$$

at the point $(1, 4)$.

Solution to 5: The answer should be an **equation of a straight line**.

First let's take the derivative:

$$\frac{dy}{dx} = 3x^2 + 2 \quad .$$

At the point $(1, 4)$, we have $x = 1$, so to get the slope at that point, we plug-in $x = 1$:

$$m = \frac{dy}{dx}\Big|_{x=1} = 3 \cdot 1^2 + 2 = 3 + 2 = 5 \quad .$$

Using the point-slope equation for a straight line, we have:

$$(y - 4) = 5(x - 1) \quad ,$$

and simplifying, we get

$$y - 4 = 5x - 5 \quad ,$$

and finally,

$$y = 5x - 1 \quad .$$

Ans. to 5: $y = 5x - 1$.

6. (12 points [3 pts each]) Find the limits

$$(a) \qquad \lim_{x \to -2} \frac{x^2 - 1}{x^2 + 2x + 3}$$

Sol. to 6a):

$$\lim_{x \to -2} \frac{x^2 - 1}{x^2 + 2x + 3} = \frac{(-2)^2 - 1}{(-2)^2 + 2 \cdot (-2) + 3} = \frac{3}{3} = 1 \quad .$$

Ans. to 6a): 1.

$$(b) \lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x}$$

Sol. to 6b:Multiplying by the **conjugate**:

$$\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x} = \lim_{x \to 0} \frac{(\sqrt{4+x} - 2)(\sqrt{4+x} + 2)}{x(\sqrt{4+x} + 2)}$$

$$= \lim_{x \to 0} \frac{(\sqrt{4+x})^2 - 2^2}{x(\sqrt{4+x} + 2)} = \lim_{x \to 0} \frac{(4+x) - 4}{x(\sqrt{4+x} + 2)}$$

$$= \lim_{x \to 0} \frac{x}{x(\sqrt{4+x} + 2)} = \lim_{x \to 0} \frac{1}{\sqrt{4+x} + 2} = \frac{1}{\sqrt{4+0} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2+2} = \frac{1}{4} \quad .$$

Ans.to 6b: $\frac{1}{4}$.

$$(c) \qquad \lim_{x \to \pi/2} \frac{1 - \cos x}{x}$$

Sol. to 6c:

$$\lim_{x \to \pi/2} \frac{1 - \cos x}{x} = \frac{1 - \cos \pi/2}{\pi/2}$$

$$= \frac{1 - 0}{\pi/2} = \frac{2}{\pi} \quad .$$

Ans.to 6c: $\frac{2}{\pi}$.

$$(d) \quad \lim_{x\to 0} \frac{\sin 10x}{\sin 5x}$$

Sol. to 6d: By **Dr. Z's get-rid of sines rule**:

$$\lim_{x\to 0} \frac{\sin 10x}{\sin 5x} = \lim_{x\to 0} \frac{10x}{5x} = \lim_{x\to 0} 2 = 2 \quad .$$

Ans.to 6d: 2.

7. (12 points) Let

$$f(x) = \begin{cases} \sqrt{-2x}, & \text{if } x < 0; \\ 3 + 2x, & \text{if } 0 \le x < 3; \\ (2x-3)^2, & \text{if } x \ge 3 \ . \end{cases}$$

(a) [1 pt. each] For each of the following limits, evaluate it, if it exists.

(i) $\lim_{x\to 0^+} f(x)$

Ans. to (i): $3 + 2 \cdot 0 = 3$.

(ii) $\lim_{x\to 0^-} f(x)$

Ans. to (ii): $\sqrt{-2 \cdot 0} = \sqrt{0} = 0$.

(iii) $\lim_{x\to 0} f(x)$

Ans. to (iii): Does Not Exist (since 0 and 3 are different).

(iv) $\lim_{x\to 3^+} f(x)$

Ans. to (iv): $(2 \cdot 3 - 3)^2 = 9$.

(v) $\lim_{x\to 3^-} f(x)$

Ans. to (v): $3 + 2 \cdot 3 = 9$.

(vi) $\lim_{x\to 3} f(x)$.

Ans. to (vi): 9 (the common value of (iv) and (v)).

(b) [6 pts] Where is f discontinuous? Explain! (no credit without explanation).

Solution to 7b): The only place where $f(x)$ is discontinuous is at $x = 0$, since there the limit of the function as x goes to 0 does not exist. In all other places the limit exists and equals to the function-value, since each piece is continuous, and the interface at $x = 3$ is also ok, since $\lim_{x\to 3}$ exists and equals $f(3)$ (both being 9).

8. (12 points) Find the point on the curve

$$y = x^2 - 3x + 2$$

where the tangent line is parallel to the line $y = x + 4$. Then find the equation of that tangent line.

Solution to 8: The slope of the line $y = x + 4$ is 1, so we have to set the derivative $\frac{dy}{dx}$ equal to 1.

First, let's find the derivative:

$$\frac{dy}{dx} = 2x - 3 \quad .$$

Setting this equal to 1

$$2x - 3 = 1 \quad ,$$

and solving for x gives

$$2x = 4 \quad ,$$

and, so

$$x = 2 \quad .$$

To get the y-coordinate, we plug-into y

$$(x^2 - 3x + 2)\Big|_{x=2} = 2^2 - 3 \cdot 2 + 2 = 4 - 6 + 2 = 0 \quad .$$

So the **point** is $(2, 0)$.

Ans. to First Part of 8: The point is $(2, 0)$.

We still need to find the equation of the tangent line. Using

$$(y - y_0) = m(x - x_0) \quad ,$$

we get

$$(y - 0) = 1 \cdot (x - 2) \quad ,$$

that yields:

$$y = x - 2 \quad .$$

Ans. to Second Part of 8: The equation of the tangent line to the curve $y = x^2 - 3x + 2$ parallel to $y = x + 4$ is $y = x - 2$.