Dr. Z's Math151 Handout #5.6 [The Substitution Rule]

By Doron Zeilberger

Problem Type 5.6.1 : Evaluate the indefinite integral

$$\int COMPLICATED(Var)\, dVar \quad .$$

Example Problem 5.6.1: Evaluate

$$\int e^x \sqrt{1+e^x}\, dx \quad .$$

Steps

1. Try to find a good u. Usually what's inside parantheses or square-root (or any root) sign is a good bet. Note that there may be more than one obvious choice, and usually only one of them works. In fact, very often, nothing works (in real life, the problems that you are likely to get, at least in this semester, should be doable!).

2. Find the derivative of u, du/dx, and then by 'cross-multiplying' express dx in terms of du.

3. Do the translation from the x language to the u language. You should eventually get ONLY u. You may need to use some algebra to express the remmaining x-stuff in terms of u, but often everything is in terms of u.

Example

1. Since $1+e^x$ is inside the square-root sign, let's try: $u = 1+e^x$.

2.

$$\frac{du}{dx} = e^x \quad hence \quad dx = \frac{du}{e^x} \quad .$$

3.

$$\int e^x \sqrt{1+e^x}\, dx \;=\; \int e^x \sqrt{u} \frac{du}{e^x} \;=\; \int \sqrt{u}\, du \quad .$$

4. Do this far simpler u integral. Finally replace u by the expression in x it stands for from step 1. Finally add $+C$.

4.

$$\int \sqrt{u}\, du = \int u^{1/2}\, du = \frac{u^{3/2}}{3/2} = (2/3)(\sqrt{u})^3 =$$

$$(2/3)(\sqrt{1+e^x})^3 + C \quad .$$

Ans.: $\frac{2}{3}(\sqrt{1+e^x})^3 + C \quad .$

Problem Type 5.6.2 : Evaluate the definite integral

$$\int_a^b COMPLICATED(Var)\, dVar \quad .$$

Example Problem 5.6.2: Evaluate

$$\int_0^{\sqrt{\pi}} x \cos(x^2)\, dx \quad .$$

Steps	**Example**
1. As before, try to find a good u.	**1.** Since x^2 is inside the cos, let's try: $u = x^2$.
2. Find the derivative of u, du/dx, and then by 'cross-multiplying' express dx in terms of du.	**2.** $$\frac{du}{dx} = 2x \quad hence \quad dx = \frac{du}{2x} \quad .$$
3. Do the translation from the x language to the u language. Now you also need to translate the *integration limits*, by plugging a and b into u. You should eventually get a *definite* integral involving ONLY u. Evaluate it!	**3.** When $x = 0$, $u = 0$. When $x = \sqrt{\pi}$, $u = \pi$. The integral equals $$\int_0^{\pi} x \cos(u) \frac{du}{2x} = \frac{1}{2} \int_0^{\pi} \cos(u)\, du =$$ $$\sin u \Big|_0^{\pi} = \sin \pi - \sin 0 = 0 - 0 = 0 \quad .$$ **Ans.**: 0.

Problem from a Previous Final (Spring 2008, #15a (7 points)) Compute

$$\int \frac{e^{1/x}}{x^2}\, dx$$

Solution: the expression "inside" the exponential is $1/x$, so a natural choice for the u is

$$u = \frac{1}{x} = x^{-1} \quad .$$

Now

$$\frac{du}{dx} = (-1)x^{-2} = \frac{-1}{x^2} \quad ,$$

and cross-multiplying, we get

$$dx = -x^2 du \quad .$$

Putting-it together

$$\int \frac{e^{1/x}}{x^2}\, dx = \int \frac{e^u}{x^2}(-x^2\, du) = -\int e^u\, du = -e^u = -e^{1/x} + C \quad .$$

Ans.: $-e^{1/x} + C$.