Dr. Z's Math151 Handout #5.4 [The Fundamental Theorem of Calculus, Part II]

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Problem Type 5.4.1 : Differentiate

$$f(Variable_1) = \int_{Number}^{Variable_1} Expression(Variable_2) d\,Variable_2 \quad .$$

Example Problem 5.4.1: Use the Fundamental Theorem of Calculus to find the derivative of

$$g(u) = \int_3^u \frac{1}{x+x^2} dx \quad .$$

Steps

1. Make sure that the argument of the function on the left side is the same as the upper limit of the integral on the right side. Also make sure that the lower limit in the integral sign is a number. Also make sure that the expression inside the integral is a function of $Variable_2$, where $Variable_2$ is the variable standing after the 'd' in $d\,Variable_2$.

Example

1. The argument of $g(u)$,u, is the same as the letter appearing at the upper limit of the integral. The lower limit is a number (3), so that's OK too. The variable of integration is x since the integral has a dx in it, and the integrand $\frac{1}{x+x^2}$ is indeed an expression of x. (If it weren't, then you would still consider it as an expression in x, viewing all other variables as silent constants.)

Steps

2. To find the derivative of $f(Variable_1)$ with respect to $Variable_1$, all you have to do is replace $Variable_2$ by $Variable_1$ in the integrand $Expression(Variable_2)$. In other words, the answer is $Expression(Variable_1)$.

Example

2. Replacing x by u in $\frac{1}{x+x^2}$ yields

Ans.: $g'(u) = \frac{1}{u+u^2}$

Problem Type 5.4.2 : Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of

$$h(Var_1) = \int_{Number}^{g(Var_1)} Expr(Var_2)\, d\,Var_2 \quad ,$$

where now the upper limit is an expression $g(Var_1)$ rather than just Var_1.

Example Problem 5.4.2: Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of

$$h(x) = \int_0^{x^2} \sqrt{1+r^3}\, dr \quad .$$

Steps

1. Here the upper limit of the integral is an expression in the variable of the named function on the left, not just the variable itself. Rewrite the derivative, using the chain rule, as

$$h'(Var_1) = \frac{d\,g(Var_1)}{d\,Var_1} \cdot \frac{d}{dg(Var_1)} \left(\int_{Number}^{g(Var_1)} Expr(Var_2) d\,Var_2 \right) \quad .$$

2. Do the differentiation $g'(Variable_1)$. The second part is done exactly as in problem 5.4.1, where you plug-in $g(Variable_1)$ in $Variable_2$.

Example

1. Here the upper limit of the integral is an expression x^2 in the variable of the named function $h(x)$, which is x on the left, not just x itself. Rewrite the derivative, using the chain rule, as

$$h'(x) = \frac{d(x^2)}{dx} \cdot \frac{d}{d\,x^2} \left(\int_0^{x^2} \sqrt{1+r^3} dr \right) \quad .$$

2. Ans.:

$$h'(x) = 2x\,\sqrt{1+(x^2)^3} = 2x\,\sqrt{1+x^6} \quad .$$

Comment: If the upper limit of integration is a number but the lower limit of integration is a variable or an expression, first rewrite it as

$$\int_{Expr(Var_1)}^{Number} = -\int_{Number}^{Expr(Var_1)} \quad ,$$

and then proceed as in 5.4.1 or 5.4.2 .

If *both* lower and upper integration limits are expressions, then first rewrite it as

$$\int_0^{Expr_2(Var_1)} - \int_0^{Expr_1(Var_1)} \quad ,$$

and do each piece separately, like in 5.4.2. Example:

$$\frac{d}{dx}\left(\int_{x^2}^{x^3} \sin u\, du\right) = \frac{d}{dx}\left(\int_0^{x^3} \sin u\, du\right) - \frac{d}{dx}\left(\int_0^{x^2} \sin u\, du\right) =$$

$$\frac{dx^3}{dx}\cdot\frac{d}{dx^3}\left(\int_0^{x^3} \sin u\, du\right) - \frac{dx^2}{dx}\cdot\frac{d}{dx^2}\left(\int_0^{x^2} \sin u\, du\right) =$$

$$3x^2 \sin x^3 - 2x \sin x^2 \quad .$$

Problem from a Previous Final (Spring 2008, #15d (7 points)) If

$$h(x) = \int_0^x t(t-2)^{30}\, dt \quad ,$$

what are the critical points of $h(x)$?

Solution: Recall that the critical points of a function $f(x)$ are the points where $f'(x) = 0$ (and where it it not defined, but this is not an issue here).

By the Fundamental Theorem of Calculus:

$$h'(x) = x(x-2)^{30} \quad .$$

Setting this equal to 0 gives the equation

$$x(x-2)^{30} = 0 \quad ,$$

whose roots are $x = 0$ and $x = 2$. So the critical points of $h(x)$ are $x = 0$ and $x = 2$.

Ans.: The critical numbers (points) of $h(x)$ are $x = 0$ and $x = 2$.