Dr. Z's Math151 Handout #5.2

Problem Type 5.2.1: Draw a graph of the signed area represented by the integral

$$\int_a^b f(x)\, dx$$

and compute it using geometry.

Example Problem 5.2.1: Draw a graph of the signed area represented by the integral

$$\int_{-2}^{1} (3x+4)\, dx$$

and compute it using geometry.

Steps	**Example**
1. Draw a picture and mark the area under the x-axis, and the area above the x-axis.	**1.** $y = 3x + 4$ is a straight line that crosses the x-axis at the point $(-4/3, 0)$. The "negative area" is the trianle with vertices $(-2, 0)$,$(-4/3, 0)$, and $(-2, -2)$. The "positive area" is the triangle with vertices $(-4/3, 0)$, $(1, 0)$, and $(1, 7)$.
2. Using Geometry (area of triangle, circles, rectangles, or trapezoids) compute the individual areas.	**2.** The triangle below the x-axis has height 2 and base $2 - 4/3 = 2/3$. Its area is $2 \cdot (2/3)/2 = 2/3$. The triangle above the x-axis has height 7 and base $1 + 4/3 = 7/3$. Its area is $7 \cdot (7/3)/2 = 49/6$.
3. Do "area above" minus "area below".	**3.** $$\frac{49}{6} - \frac{2}{3} = \frac{49}{6} - \frac{4}{6} = \frac{45}{6} = \frac{15}{2} \quad .$$ **Ans.**: $\frac{15}{2}$.

Problem Type 5.2.2: Use graphical consideration to compute the integral

$$\int_{-a}^{a} Complicated(x) \quad ,$$

where $Complicated(x)$ happens to be an odd function.

(recall that an odd function is a function $f(x)$ such that $f(-x)$ is the same as $-f(x)$

Example Problem 5.2.2: Use graphical consideration to compute the integral

$$\int_{-2}^{2} (\tan(x^3) + 3x)\, dx \quad ,$$

Steps

1. In order for this to work, the limits of integration of the definite integral have to be of the form $\int_{-a}^{a}$. For example, if you were asked to do

$$\int_{-1}^{2} (\tan(x^3) + 3x)\, dx \quad ,$$

there is no way it can be done with "geometrical considerations". Also check that the function being integrated (the integrand) is odd, by plugging in $-x$ to x and simplifying. You may have to use simple algebra and trig.

2. Now you argue as follows. Since the function is odd, the region between $x = -a$ and the y axis is opposite to the region between the y-axis and $x = a$. So the "positive area" is the same as the "negative area", and their difference is 0.

Example

1. The integral is indeed of the form $\int_{-a}^{a} f(x)$ (here $a = -2$). Now for $f(x) = \tan(x^3) + 3x$, we have

$$f(-x) = \tan((-x)^3) + 3(-x) = \tan(-x^3) - 3x$$

$$= -\tan(x^3) - 3x = -(\tan(x^3) + 3x) = -f(x) \quad .$$

(recall that *tan* is an odd function: $\tan(-w) = -\tan w$ (since sin is odd and cos is even)).

2. Now you argue as follows. Since the function is odd, the region between $x = -2$ and the y axis is opposite to the region between the y-axis and $x = 2$. So the "positive area" is the same as the "negative area", and their difference is 0.

Ans. 0.

A problem from a previous Final Exam (Spring 2008, #15(c) (7 points))

Find

$$\int_0^5 \sqrt{25-x^2}\, dx \quad .$$

Comment: In Calc II, you will learn how to do this kind of integrals directly, but in Calc I, the only way you can do it is via "geometrical consideration". Note that in this problem you were not asked to use "geometrical consideration". Nevertheless, you are supposed to figure out by yourself that this is the only way that **you** can do it. Even if you have already taken Calc II before, and know how to do it the "hard way", it is still better to use Geometry!

Solution: This integral represents the area under $y = \sqrt{25-x^2}$ above $0 \leq x \leq 5$. But this is nothing but a quarter-circle of radius 5, so its area is

$$\frac{\pi \cdot 5^2}{4} = \frac{25\pi}{4} \quad .$$

Ans.: $\frac{25\pi}{4}$.

Another problem from a previous Final Exam (Spring 2008, #13(c) (8 points))

Using graphic analysis, find the definite integral

$$\int_{-1}^{1} \frac{\tan x - x}{x^2+3}\, dx$$

(You must justify your answer).

Solution. The function being integrated

$$f(x) = \frac{\tan x - x}{x^2+3}$$

is an odd function, since

$$f(-x) = \frac{\tan(-x)+x}{(-x)^2+3} = \frac{-\tan(x)+x}{x^2+3} = \frac{-(\tan(x)-x)}{x^2+3} = -f(x) \quad .$$

The definite integral represents the signed area between $x = -1$ and $x = 1$ and the curve $y = \frac{\tan x - x}{x^2+3}$. Since the function is odd, the area above the x-axis is the same as the area below the x-axis, so their difference is 0.

Ans. 0.