Problem Type 4.9.1: Find the most general antiderivative of the function \( f(x) \).

**Example Problem 4.9.1**: Find the most general antiderivative of the function \( f(x) = 5e^x + 8\sec^2 x \).

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<td>1. You have to memorize the differentiation table in reverse. An antiderivative of ( x^n ) if ( x^{n+1}/(n+1) ) (except when ( n = -1 )). The antiderivative of ( \cos x ) is ( \sin x ), The antiderivative of ( \sin x ) is ( -\cos x ), The antiderivative of ( \sec^2 x ) is ( \tan x ), etc.</td>
<td>1. An antiderivative of ( e^x ) is ( e^x ), and that of ( \sec^2 x ) is ( \tan x ). Hence an antiderivative of ( f(x) ) is ( 5e^x + 8\tan x ).</td>
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<td>2. To find the most general antiderivative, you add ( C ) (an arbitrary constant) to the above answer.</td>
<td>2. Ans.: ( 5e^x + 8\tan x + C ).</td>
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Problem Type 4.9.2: Find \( f \) if \( f'(x) = \text{Expression}(x) \) and \( f(a) = \text{Number} \).

Example Problem 4.9.2: Find \( f \) if \( f'(x) = 6x - 2/x^2, x > 0 \) and \( f(1) = 4 \).

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<td>1. Find the most general antiderivative, like in problem 4.9.1, featuring ( C ).</td>
<td>1. ( f'(x) = 6x - 2/x^2 ) so ( f(x) = 3x^2 + 2x^{-1} + C )</td>
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<tr>
<td>2. Plug-in ( x = a ) and solve for ( C ) the equation ( f(a) = \text{Number} ).</td>
<td>2. ( f(x) = 3x^2 + 2x^{-1} + C ), so ( f(1) = 3 \cdot 1^2 + 2/1 + C = 5 + C ). But, on the other hand, from the data, ( f(1) = 4 ). So we have the equation ( 5 + C = 4 ). Solving it yields ( C = -1 ).</td>
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<td>3. Plug-in the specific ( C ) that you got in step 2 into the answer of step 1.</td>
<td>3. Ans.: ( f(x) = 3x^2 + 2x^{-1} - 1 ).</td>
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A problem from a Previous Final (Spring 2008, #8 (10 points)).

Find \( y = y(x) \) if \( \frac{d^2y}{dx^2} = 4x, \frac{dy}{dx}(0) = 1 \) and \( y(0) = 0 \).

Solution:

We know that \( y'' = 4x \). To get \( y' \), we take the anti-derivative of \( y'' \)

\[
y' = \int 4x \, dx = 4 \frac{x^2}{2} = 2x^2 + C .
\]

Next we need to find the value of \( C \). Plugging-in \( x = 0 \), we get, since the problem tells us that \( y'(0) = 1 \)

\[
1 = 2 \cdot 0^2 + C
\]

which is

\[
1 = C
\]

and this gives \( C = 1 \).

Intermediate answer: \( y'(x) = 2x^2 + 1 \).
To find \( y(x) \), our ultimate goal, we find the **anti-derivative** of \( 2x^2 + 1 \).

\[
y(x) = \int (2x^2 + 1) \, dx = \frac{2x^3}{3} + x + C = \frac{2x^3}{3} + x + C ,
\]

(note that this \( C \) is a **different** \( C \) than the one above).

To find this new \( C \) we plug-in \( x = 0 \). The problem tells us that \( y(0) = 0 \), so

\[
0 = \frac{2 \cdot 0^3}{3} + 0 + C
\]

which means

\[
0 = C .
\]

So \( C = 0 \). Going back, above we have

\[
y(x) = \frac{2x^3}{3} + x + 0 = \frac{2x^3}{3} + x .
\]

**Final Ans.:** \( y(x) = \frac{2x^3}{3} + x . \)