Dr. Z's Math151 Handout #4.5 [Graph Sketching and Asymptotes]

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Problem Type 4.5.1 : Sketch the curve $y = Polynomial(x)$.

Example Problem 4.5.1: Sketch the curve $y = x^4 + 4x^3$

Steps	**Example**
1. Find the first and second derivatives dy/dx and d^2y/dx^2.	**1.** $y = x^4 + 4x^3$, $dy/dx = 4x^3 + 12x^2 = 4x^2(x+3)$, $d^2y/dx^2 = 12x^2 + 24x = 12x(x+2)$.
2. Find the critical numbers by solving $dy/dx = 0$ and the inflection points by solving $d^2y/dx^2 = 0$.	**2.** Critical numbers are solutions of $4x^3 + 12x^2 = 0$, which means $4x^2(x+3) = 0$, whose roots are $x = -3$ and $x = 0$. The (potential) inflection points are when $12x^2 + 24x = 0$, which means $12x(x+2)$ whose solutions are $x = -2$ and $x = 0$. Combining we have the **important numbers**: $x = -3, x = -2, x = 0$.
3. Plug the important numbers into y to get the corresponding y-coordinates, thereby finding the **important points**.	**3.** $y = x^4 + 4x^3 = x^3(x+4)$, When $x = -3$, $y = (-3)^3 \cdot 1 = -27$. When $x = -2$, $y = (-2)^3 \cdot 2 = -16$. When $x = 0$, $y = (0)^3 \cdot 4 = 0$. **Important points**: $(-3, -27)$, $(-2, -16)$, $(0, 0)$.

4. Use the second derivative test to find which of the critical numbers are local max (if the 2nd derivative is negative) , min (if the 2nd derivative is positive) or don't know (if it 0).

4.

When $x = -3$ $d^2y/dx^2 = 12(-3)(-3+2) > 0$, so $x = -3$ is a local min.

When $x = 0$ $d^2y/dx^2 = 0$, so it is inconclusive. In that case you see whether the first-derivative changes sign. In this problem it is positive on both sides, so $x = 0$ is neither! So $(0, 0)$ is **horizontal inflection point**.

5. Summarize your finding of the important points, and their claim to fame.

5. $(-3, -27)$: local min. $(-2, -16)$: inflection pt. $(0, 0)$: horizontal inflection point.

6. Mark the important points on the $x-y$ plane, drawing a little cup at minima, and a little cap at maxima, and a short underline at horiz. inflection points. If you wish, you can also compute the slope at the inflection point, and draw a tiny line-segment with that slope there. Then 'connect the dots' to sketch the graph.

6.

Problem Type 4.5.2 : Sketch the curve $y = Rational(x)$.

Example Problem 4.5.2: Sketch the curve $y = \frac{x}{x^2-9}$.

Steps

1. Find the horizontal asymptotes by taking the limit as $x \to \infty$. Find the vertical asymptotes by setting the denominator equal to zero.

2. Find dy/dx and d^2y/dx^2; Find the critical numbers (in addition to the vertical asymptotes) by setting $dy/dx = 0$ and determine whether each is a max or min according to the second derivative test. Find the inflection points by solving $d^2y/dx^2 = 0$.

3. For each vertical asymptote $x = a$, find $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$, each of which is ∞ or $-\infty$. This will tell you about the behavior before and after each vertical asymptote.

Example

1. Horizontal asymptote:

$$y = \lim_{x\to\infty} \frac{x}{x^2-9} = 0 \quad ,$$

Vertical asymptote: solving $x^2 - 9 = 0$ gives $x = -3$ and $x = 3$.

2.

$$\frac{dy}{dx} = -\frac{x^2+9}{(x^2-9)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2x(x^2+27)}{(x^2-9)^3}$$

critical numbers: solving $dy/dx = 0$ yields the equation $x^2 + 9 = 0$, that has no real roots. So there are no local max or min.

inflection point: $x = 0$. When $x = 0$, $y = 0$, so the inflection point is $(0, 0)$. The slope there is $-1/9$.

3. $y = x/((x-3)(x+3)$ is negative when $x < -3$ so $\lim_{x\to -3^-} f(x) = -\infty$ and the curve shoots down before $x = -3$. y is positive for x between -3 and 0, so the curve comes down from ∞ after $x = -3$. y is negative before $x = 3$, so shoots down to $-\infty$ at $x = 3$, and it is positive for $x > 3$, so it comes down from ∞ at $x = 3$.

A Problem from a Previous Final (Spring 2008, #9 (20 points))

Let

$$f(x) = \frac{3}{4 - x^2} \quad .$$

a) (5 points) Find the horizontal and vertical asymptotes of the graph of $f(x)$.

b) (5 points) Determine the intervals on which f is increasing and decreasing, and the local extrema.

c) (4 points) Determine the intervals on which f is concave up or concave down, and inflection points

d) (6 points) Sketch the graph of $f(x)$.

Solution

a) To get the horizontal asymptotes we do

$$\lim_{x \to +\infty} \frac{3}{4 - x^2} = 0$$

$$\lim_{x \to -\infty} \frac{3}{4 - x^2} = 0$$

So $y = 0$ (the x-axis) is a horizontal asymptote on both sides.

To get the vertical asymptotes we set the denominator equal to 0

$$4 - x^2 = 0 \quad .$$

Solving we get

$$-(x - 2)(x + 2) = 0 \quad ,$$

and the solutions are $x = -2$ and $x = 2$.

Ans. to a): The horiz. asymptotes are $y = 0$ (alias the x-axis) (on either side), and the vertical asymptotes are $x = -2$ and $x = 2$.

b) Using either the quotient rule or the chain rule $f(x) = 3(4 - x^2)^{-1}$, we get

$$f'(x) = \frac{6x}{(4 - x^2)^2} \quad .$$

Solving $f'(x) = 0$ is the same as solving $6x = 0$ so $x = 0$ is a **local extreme point** (we still don't know whether it is max or min), this will come up later.

Anyway, now we have the **important numbers** $x = -2$, $x = 0$, $x = 2$, that divide the x-axis into **four** intervals, for each of which we have to decide whether the function is increasing or decreasing, by plugging-in some random point.

First interval: $(-\infty, -2)$. A good point to pick is $x = -3$. Plugging $x = -3$ into $f'(x)$ we get

$$f'(-3) = \frac{6 \cdot (-3)}{(4 - (-3))^2} < 0$$

Since this is negative we conclude that the function is **decreasing** in the interval $(-\infty, -2)$.

Second interval: $(-2, 0)$. A good point to pick is $x = -1$. Plugging $x = -1$ into $f'(x)$ we get

$$f'(-1) = \frac{6 \cdot (-1)}{(4 - (-1))^2} < 0$$

Since this is negative we conclude that the function is **decreasing** in the interval $(-2, 0)$.

Third interval: $(0, 2)$. A good point to pick is $x = 1$. Plugging $x = 1$ into $f'(x)$ we get

$$f'(1) = \frac{6 \cdot (1)}{(4 - (1))^2} > 0$$

Since this is positive we conclude that the function is **increasing** in the interval $(0, 2)$.

Fourth interval: $(2, \infty)$. A good point to pick is $x = 3$. Plugging $x = 3$ into $f'(x)$ we get

$$f'(3) = \frac{6 \cdot (3)}{(4 - (3))^2} > 0$$

Since this is positive we conclude that the function is **increasing** in the interval $(2, \infty)$.

Also since $x = 0$ is a transition between decreasing and increasing, we know that $x = 0$ is a local minimum. Since $f(0) = 3/4$, we get that the point $(0, 3/4)$ is a local minimum point.

Ans. to b): The function is decreasing in the intervals $(-\infty, -2)$ and $(-2, 0)$ and increasing in the intervals $(0, 2)$ and $(2, \infty)$. $x = 0$ (with the corresponding point $(0, 4/3)$ is a local minimum point.

c) It is a bit of a pain to find $f''(x)$, but it turns out to be:

$$f''(x) = \frac{6(3x^2 + 4)}{(4 - x^2)^3}$$

(you do it!). To get the inflection points we set the top equal to 0 and get the equation $3x^2 + 4 = 0$, that has **no** solution. So there are no inflection points. To determine the intervals of concave-up and concave-down, we need only consider the division of the x-axis induced by the two "blow-up" points, $x = -2$ and $x = 2$.

First Interval: $(-\infty, -2)$ Pick a random point, say $x = -3$ and get

$$f''(-3) = \frac{6 \cdot (-3)^2 + 4)}{(4 - (-3)^2)^3} < 0$$

since it is negative we have that in the interval $(-\infty, -2)$ our function is **concave-down**.

Second Interval: $(-2, 2)$ Pick a random point, say $x = 0$ and get

$$f''(0) = \frac{6 \cdot (0)^2 + 4)}{(4 - (0)^2)^3} > 0$$

since it is positive we have that in the interval $(-2, 2)$ our function is **concave-up**.

Third Interval: $(2, \infty)$ Pick a random point, say $x = 3$ and get

$$f''(3) = \frac{6 \cdot (3)^2 + 4)}{(4 - (3)^2)^3} < 0$$

since it is negative we have that in the interval $(2, \infty)$ our function is **concave-down**.

Ans. to c): The function $f(x)$ is concave-down in the infinite intervals $(-\infty, -2)$ and $(2, \infty)$, and it is concave-up in the interval $(-2, 2)$. There are **no inflection points**.

d)