Problem Type 4.5.1: Sketch the curve \( y = \text{Polynomial}(x) \).

Example Problem 4.5.1: Sketch the curve \( y = x^4 + 4x^3 \)

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Steps

1. Find the first and second derivatives \( dy/dx \) and \( d^2y/dx^2 \).

\[
\begin{align*}
    y &= x^4 + 4x^3, \\
    dy/dx &= 4x^3 + 12x^2, \\
    d^2y/dx^2 &= 12x^2 + 24x.
\end{align*}
\]

2. Find the critical numbers by solving \( dy/dx = 0 \) and the inflection points by solving \( d^2y/dx^2 = 0 \).

Critical numbers are solutions of \( 4x^3 + 12x^2 = 0 \), which means \( 4x^2(x + 3) = 0 \), whose roots are \( x = -3 \) and \( x = 0 \). The (potential) inflection points are when \( 12x^2 + 24x = 0 \), which means \( 12x(x + 2) \) whose solutions are \( x = -2 \) and \( x = 0 \). Combining we have the important numbers: \( x = -3, x = -2, x = 0 \).

3. Plug the important numbers into \( y \) to get the corresponding \( y \)-coordinates, thereby finding the important points.

\[
\begin{align*}
    y &= x^4 + 4x^3 = x^3(x + 4), \\
\end{align*}
\]

When \( x = -3 \), \( y = (-3)^3 \cdot 1 = -27 \).

When \( x = -2 \), \( y = (-2)^3 \cdot 2 = -16 \).

When \( x = 0 \), \( y = (0)^3 \cdot 4 = 0 \).

Important points: \((−3, −27), (−2, −16), (0, 0)\).
4. Use the second derivative test to find which of the critical numbers are local max (if the 2nd derivative is negative), min (if the 2nd derivative is positive) or don’t know (if it 0).

When \( x = -3 \) \( \frac{d^2 y}{dx^2} = 12(-3)(-3 + 2) > 0 \), so \( x = -3 \) is a local min.

When \( x = 0 \) \( \frac{d^2 y}{dx^2} = 0 \), so it is inconclusive. In that case you see whether the first-derivative changes sign. In this problem it is positive on both sides, so \( x = 0 \) is neither! So \((0,0)\) is horizontal inflection point.

5. Summarize your finding of the important points, and their claim to fame.

\( (-3, -27) \): local min. \( (-2, -16) \): inflection pt. \((0,0)\): horizontal inflection point.

6. Mark the important points on the \( x - y \) plane, drawing a little cup at minima, and a little cap at maxima, and a short underline at horiz. inflection points. If you wish, you can also compute the slope at the inflection point, and draw a tiny line-segment with that slope there. Then ‘connect the dots’ to sketch the graph.
Problem Type 4.5.2: Sketch the curve \( y = \text{Rational}(x) \).

Example Problem 4.5.2: Sketch the curve \( y = \frac{x}{x^2 - 9} \).

Steps

1. Find the horizontal asymptotes by taking the limit as \( x \to \infty \). Find the vertical asymptotes by setting the denominator equal to zero.

Example

1. Horizontal asymptote:

\[
 y = \lim_{x \to \infty} \frac{x}{x^2 - 9} = 0
\]

Vertical asymptote: solving \( x^2 - 9 = 0 \) gives \( x = -3 \) and \( x = 3 \).

2. Find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \); Find the critical numbers (in addition to the vertical asymptotes) by setting \( \frac{dy}{dx} = 0 \) and determine whether each is a max or min according to the second derivative test. Find the inflection points by solving \( \frac{d^2y}{dx^2} = 0 \).

Example

2.\[
 \frac{dy}{dx} = -\frac{x^2 + 9}{(x^2 - 9)^2}
\]
\[
 \frac{d^2y}{dx^2} = \frac{2x(x^2 + 27)}{(x^2 - 9)^3}
\]

Critical numbers: solving \( \frac{dy}{dx} = 0 \) yields the equation \( x^2 + 9 = 0 \), that has no real roots. So there are no local max or min.

Inflection point: \( x = 0 \). When \( x = 0 \), \( y = 0 \), so the inflection point is \((0, 0)\). The slope there is \(-1/9\).

3. For each vertical asymptote \( x = a \), find \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \), each of which is \( \infty \) or \( -\infty \). This will tell you about the behavior before and after each vertical asymptote.

Example

3. \( y = \frac{x}{((x - 3)(x + 3))} \) is negative when \( x < -3 \) so \( \lim_{x \to -3^-} f(x) = -\infty \) and the curve shoots down before \( x = -3 \). \( y \) is positive for \( x \) between \(-3 \) and \( 0 \), so the curve comes down from \( \infty \) after \( x = -3 \). \( y \) is negative before \( x = 3 \), so shoots down to \( -\infty \) at \( x = 3 \), and it is positive for \( x > 3 \), so it comes down from \( \infty \) at \( x = 3 \).
A Problem from a Previous Final (Spring 2008, #9 (20 points))

Let

\[ f(x) = \frac{3}{4 - x^2} \]

a) (5 points) Find the horizontal and vertical asymptotes of the graph of \( f(x) \).

b) (5 points) Determine the intervals on which \( f \) is increasing and decreasing, and the local extrema.

c) (4 points) Determine the intervals on which \( f \) is concave up or concave down, and inflection points

d) (6 points) Sketch the graph of \( f(x) \).

Solution

a) To get the horizontal asymptotes we do

\[ \lim_{x \to +\infty} \frac{3}{4 - x^2} = 0 \]
\[ \lim_{x \to -\infty} \frac{3}{4 - x^2} = 0 \]

So \( y = 0 \) (the \( x \)-axis) is a horizontal asymptote on both sides.

To get the vertical asymptotes we set the denominator equal to 0

\[ 4 - x^2 = 0 \]

Solving we get

\[ -(x - 2)(x + 2) = 0 \]

and the solutions are \( x = -2 \) and \( x = 2 \).

Ans. to a): The horiz. asymptotes are \( y = 0 \) (alias the \( x \)-axis) (on either side), and the vertical asymptotes are \( x = -2 \) and \( x = 2 \).

b) Using either the quotient rule or the chain rule \( f(x) = 3(4 - x^2)^{-1} \), we get

\[ f'(x) = \frac{6x}{(4 - x^2)^2} \]

Solving \( f'(x) = 0 \) is the same as solving \( 6x = 0 \) so \( x = 0 \) is a local extreme point (we still don’t know whether it is max or min), this will come up later.

Anyway, now we have the important numbers \( x = -2, x = 0, x = 2 \), that divide the \( x \)-axis into four intervals, for each of which we have to decide whether the function is increasing or decreasing, by plugging-in some random point.
**First interval**: $(-\infty, -2)$. A good point to pick is $x = -3$. Plugging $x = -3$ into $f'(x)$ we get

$$f'(-3) = \frac{6 \cdot (-3)}{(4 - (-3))^2} < 0$$

Since this is negative we conclude that the function is **decreasing** in the interval $(-\infty, -2)$.

**Second interval**: $(-2, 0)$. A good point to pick is $x = -1$. Plugging $x = -1$ into $f'(x)$ we get

$$f'(-1) = \frac{6 \cdot (-1)}{(4 - (-1))^2} < 0$$

Since this is negative we conclude that the function is **decreasing** in the interval $(-2, 0)$.

**Third interval**: $(0, 2)$. A good point to pick is $x = 1$. Plugging $x = 1$ into $f'(x)$ we get

$$f'(1) = \frac{6 \cdot (1)}{(4 - (1))^2} > 0$$

Since this is positive we conclude that the function is **increasing** in the interval $(0, 2)$.

**Fourth interval**: $(2, \infty)$. A good point to pick is $x = 3$. Plugging $x = 3$ into $f'(x)$ we get

$$f'(3) = \frac{6 \cdot (3)}{(4 - (3))^2} > 0$$

Since this is positive we conclude that the function is **increasing** in the interval $(2, \infty)$.

Also since $x = 0$ is a transition between decreasing and increasing, we know that $x = 0$ is a local minimum. Since $f(0) = 3/4$, we get that the point $(0, 3/4)$ is a local minimum point.

**Ans. to b)**: The function is decreasing in the intervals $(-\infty, -2)$ and $(-2, 0)$ and increasing in the intervals $(0, 2)$ and $(2, \infty)$. $x = 0$ (with the corresponding point $(0, 4/3)$) is a local minimum point.

**c)** It is a bit of a pain to find $f''(x)$, but it turns out to be:

$$f''(x) = \frac{6(3x^2 + 4)}{(4 - x^2)^3}$$

(you do it!). To get the inflection points we set the top equal to 0 and get the equation $3x^2 + 4 = 0$, that has no solution. So there are no inflection points. To determine the intervals of concave-up and concave-down, we need only consider the division of the $x$-axis induced by the two “blow-up” points, $x = -2$ and $x = 2$.

**First Interval**: $(-\infty, -2)$ Pick a random point, say $x = -3$ and get

$$f''(-3) = \frac{6 \cdot (-3)^2 + 4}{(4 - (-3)^2)^3} < 0$$
since it is negative we have that in the interval \((-\infty, -2)\) our function is **concave-down**.

**Second Interval:** \((-2, 2)\) Pick a random point, say \(x = 0\) and get

\[
 f''(0) = \frac{6 \cdot (0)^2 + 4}{(4 - (0)^2)^3} > 0
\]

since it is positive we have that in the interval \((-2, 2)\) our function is **concave-up**.

**Third Interval:** \((2, \infty)\) Pick a random point, say \(x = 3\) and get

\[
 f''(3) = \frac{6 \cdot (3)^2 + 4}{(4 - (3)^2)^3} < 0
\]

since it is negative we have that in the interval \((2, \infty)\) our function is **concave-down**.

**Ans. to c):** The function \(f(x)\) is concave-down in the infinite intervals \((-\infty, -2)\) and \((2, \infty)\), and it is concave-up in the interval \((-2, 2)\). There are **no inflection points**.

d)