Dr. Z's Math151 Handout #4.4 [The Shape of a Graph]

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Problem Type 4.4.1 : Given a function $f(x) = Polynomial(x)$,

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the local maximum and minimum values of f.

(c) Find the intervals of concavity and the inflection points.

Example Problem 4.4.1: As above with $f(x) = 5 - 3x^2 + x^3$.

Steps

a. Find $f'(x)$ and find the critical numbers by solving $f'(x) = 0$. If applicable, add the 'blow-up' critical numbers. Calling the critical numbers (in increasing order) $a_1, \ldots, a_k$, etc. this naturally breaks up the real line into the **consistent intervals** $(-\infty, a_1)$,(a_1, a_2), ..., (a_k, ∞). For each of these intervals pick a **convenient** point and plug into $f'(x)$ to determine the sign. If $f'(x)$ happens to be **positive** then the function $f(x)$ is **increasing** in that interval. If $f'(x)$ happens to be **negative** then the function $f(x)$ is **decreasing** in that interval.

Example

a.

$$f'(x) = (5 - 3x^2 + x^3)' =$$

$$-6x + 3x^2 = 3x(x - 2)$$

Solving $3x(x - 2) = 0$ gives the critical numbers $x = 0$ and $x = 2$. This divides the real line into the three intervals

(i) $(-\infty, 0)$. Picking, for example, $x = -1$ and plugging-in $f'(x)$ gives $f'(-1) = (-3)(-3) = 9$ which is **positive**, hence

f is **increasing** on $(-\infty, 0)$.

(ii) $(0, 2)$ picking, for example, $x = 1$ and plugging-in $f'(x)$ gives $f'(1) = (3)(-1) = -3$ which is **negative**, hence

f is **decreasing** on $(0, 2)$.

(iii) $(2, \infty)$. Picking, for example, $x = 3$ and plugging-in $f'(3)$ gives $f'(3) = 9$ which is **positive**, hence

f is **increasing** on $(2, \infty)$.

b. For each of the critical numbers, look at the interval to its left, and the interval to its right. If that number is a **transition from increasing to decreasing** it is a **local maximum**. If that number is a **transition from decreasing to increasing** it is a **local minimum**. If it is not a transition point, i.e. both intervals have the same behavior, then it is **neither**.

To get the **local minimum value** and **local maximum value** plut the local minimum and local maximum points into $f(x)$.

b.

$x = 0$ is the border-number between (i) $(-\infty, 0)$: increasing to (ii) $(0, 2)$: decreasing. Hence

$x = 0$ is a local maximum.

The **local maximum value** is $f(0) = 5$. Hence the point $(0, 5)$ is a peak.

$x = 2$ is the border-number between (ii) $(0, 2)$: decreasing to (ii) $(2, \infty)$: increasing. Hence

$x = 2$ is a local minimum.

The **local minumum value** is $f(2) = 5 - 3 \cdot 2^2 + 2^3 = 1$. Hence the point $(2, 1)$ is a valley.

c. Find $f''(x)$ and find the (potential) inflection points by solving $f''(x) = 0$. If applicable, add the 'blow-up' critical numbers. Ordering all these numbers $b_1, \ldots, b_r$, etc. this naturally breaks up the real line into the **consistent intervals** (regarding convexity) $(-\infty, b_1)$,(b_1, b_2), ..., (b_r, ∞). For each of these intervals pick a **convenient** point and plug into $f''(x)$ to determine the sign. If $f''(x)$ happens to be **positive** then the function $f(x)$ is **concave up** in that interval. If $f''(x)$ happens to be **negative** then the function $f(x)$ is **concave down** in that interval. The transition point(s) between concave-up and concave-down (or vice versa) are the **inflection points**.

c.

$$f''(x) = (3x^2 - 6x)' = 6x - 6 \quad .$$

Solving $6x - 6 = 0$ gives $x = 1$. This divides the real line into two intervals

(i) $(-\infty, 1)$. Pick, say, $x = 0$, and evaluate $f''(0) = -6$ which is **negative**, hence in this interval f **concave down**.

(ii) $(1, \infty)$. Pick, say, $x = 2$, and evaluate $f''(2) = 6$ which is **positive**, hence in this interval f is **concave up**.

The only inflection point is at $x = 1$. The corresponding y value is $f(1) = 5 - 3 \cdot 1^2 + 1^3 = 3$. So the inflection point is $(1, 3)$.

Summary: the function $f(x)$ is increasing in the intervals $(-\infty, 0)$ and $(2, \infty)$, decreasing in the interval $(0, 2)$. The local maximum is at the point $(0, 5)$ and the local minimum is $(2, 1)$. It is concave down in the interval $(-\infty, 1)$ and concave up in the interval $(1, \infty)$. The inflection point is at $(1, 3)$.

Problem Type 4.4.2: Find the critical points of $f(x)$ and use the Second Derivative Test (if possible) to determine whether each corresponds to a local minimum or maximum.

Example Problem 4.4.2: Find the critical points of $f(x) = 3x^5 - 5x^3$ and use the Second Derivative Test (if possible) to determine whether each corresponds to a local minimum or maximum.

Steps

1. Compute $f'(x)$ and $f''(x)$.

Example

1.

$$f'(x) = (3x^5 - 5x^3)' = 15x^4 - 15x^2 \quad ,$$

$$f''(x) = (15x^4 - 15x^2)' = 60x^3 - 30x \quad .$$

2. Find the critical points by solving $f'(x) = 0$.

2.

$$15x^4 - 15x^2 = 0$$

means

$$15x^2(1 - x^2) = 0$$

whose roots are $x = -1, x = 0, x = 1$.

3. For each of the critical points, plug-them-in $f''(x)$. If it is **positive** then it is a **local minimum**. If it is **negative**, then it is a **local maximum**. If it is **zero**, then **you don't know**.

3.

$$f''(-1) = 60 \cdot (-1)^3 - 30 \cdot (-1) = -30 \quad ,$$

is **negative**, so $x = -1$ is a **local maximum**.

$$f''(0) = 60 \cdot (0)^3 - 30 \cdot (0) = 0 \quad ,$$

is **zero**, so for $x = 0$ **we don't know**

$$f''(1) = 60 \cdot (1)^3 - 30 \cdot (1) = 30 \quad ,$$

is **positive**, so $x = 1$ is a **local minimum**.

Comment: By the **First Derivative Test** $f'(x)$ does not change sign when it passes $x = 0$, so it is neither max. nor min., but a **horizontal inflection point**.