Problem Type 4.4.1: Given a function \( f(x) = \text{Polynomial}(x) \),

(a) Find the intervals on which \( f \) is increasing or decreasing.

(b) Find the local maximum and minimum values of \( f \).

(c) Find the intervals of concavity and the inflection points.

Example Problem 4.4.1: As above with \( f(x) = 5 - 3x^2 + x^3 \).

Steps

**a.** Find \( f'(x) \) and find the critical numbers by solving \( f'(x) = 0 \). If applicable, add the ‘blow-up’ critical numbers. Calling the critical numbers (in increasing order) \( a_1, \ldots, a_k \), etc. this naturally breaks up the real line into the consistent intervals \((-\infty, a_1), (a_1, a_2), \ldots, (a_k, \infty)\). For each of these intervals pick a convenient point and plug into \( f'(x) \) to determine the sign. If \( f'(x) \) happens to be positive then the function \( f(x) \) is increasing in that interval. If \( f'(x) \) happens to be negative then the function \( f(x) \) is decreasing in that interval.

Example

**a.**

\[
\begin{align*}
f'(x) &= (5 - 3x^2 + x^3)' = \\
&= -6x + 3x^2 = 3x(x - 2)
\end{align*}
\]

Solving \( 3x(x - 2) = 0 \) gives the critical numbers \( x = 0 \) and \( x = 2 \). This divides the real line into the three intervals

(i) \((-\infty, 0)\). Picking, for example, \( x = -1 \) and plugging-in \( f'(x) \) gives \( f'(-1) = (-3)(-3) = 9 \) which is positive, hence \( f \) is increasing on \((-\infty, 0)\).

(ii) \((0, 2)\) picking, for example, \( x = 1 \) and plugging-in \( f'(x) \) gives \( f'(1) = (3)(-1) = -3 \) which is negative, hence \( f \) is decreasing on \((0, 2)\).

(iii) \((2, \infty)\). Picking, for example, \( x = 3 \) and plugging-in \( f'(3) \) gives \( f'(3) = 9 \) which is positive, hence \( f \) is increasing on \((2, \infty)\).
b. For each of the critical numbers, look at the interval to its left, and the interval to its right. If that number is a transition from increasing to decreasing it is a local maximum. If that number is a transition from decreasing to increasing it is a local minimum. If it is not a transition point, i.e. both intervals have the same behavior, then it is neither.

To get the local minimum value and local maximum value put the local minimum and local maximum points into $f(x)$.

\[ x = 0 \] is the border-number between (i) $(-\infty, 0)$: increasing to (ii) $(0, 2)$: decreasing. Hence

\[ x = 0 \] is a local maximum.

The local maximum value is $f(0) = 5$. Hence the point $(0, 5)$ is a peak.

\[ x = 2 \] is the border-number between (ii) $(0, 2)$: decreasing to (ii) $(2, \infty)$: increasing. Hence

\[ x = 2 \] is a local minimum.

The local minimum value is $f(2) = 5 - 3 \cdot 2^2 + 2^3 = 1$. Hence the point $(2, 1)$ is a valley.
c. Find $f''(x)$ and find the (potential) inflection points by solving $f''(x) = 0$. If applicable, add the ‘blow-up’ critical numbers. Ordering all these numbers $b_1, \ldots, b_r$, etc. this naturally breaks up the real line into the consistent intervals (regarding convexity) $(-\infty, b_1), (b_1, b_2), \ldots, (b_r, \infty)$. For each of these intervals pick a convenient point and plug into $f''(x)$ to determine the sign. If $f''(x)$ happens to be positive then the function $f(x)$ is concave up in that interval. If $f''(x)$ happens to be negative then the function $f(x)$ is concave down in that interval. The transition point(s) between concave-up and concave-down (or vice versa) are the inflection points.

c. 

$$f''(x) = (3x^2 - 6x)' = 6x - 6$$ 

Solving $6x - 6 = 0$ gives $x = 1$. This divides the real line into two intervals 

(i) $(-\infty, 1)$. Pick, say, $x = 0$, and evaluate $f''(0) = -6$ which is negative, hence in this interval $f$ concave down.

(ii) $(1, \infty)$. Pick, say, $x = 2$, and evaluate $f''(2) = 6$ which is positive, hence in this interval $f$ is concave up.

The only inflection point is at $x = 1$. The corresponding $y$ value is $f(1) = 5 - 3 \cdot 1^2 + 1^3 = 3$. So the inflection point is $(1, 3)$.

**Summary:** the function $f(x)$ is increasing in the intervals $(-\infty, 0)$ and $(2, \infty)$, decreasing in the interval $(0, 2)$. The local maximum is at the point $(0, 5)$ and the local minimum is $(2, 1)$. It is concave down in the interval $(-\infty, 1)$ and concave up in the interval $(1, \infty)$. The inflection point is at $(1, 3)$.

**Problem Type 4.4.2:** Find the critical points of $f(x)$ and use the Second Derivative Test (if possible) to determine whether each corresponds to a local minimum or maximum.

**Example Problem 4.4.2:** Find the critical points of $f(x) = 3x^5 - 5x^3$ and use the Second Derivative Test (if possible) to determine whether each corresponds to a local minimum or maximum.

**Steps**

1. Compute $f'(x)$ and $f''(x)$.

**Example**

1. 

$$f'(x) = (3x^5 - 5x^3)' = 15x^4 - 15x^2$$

$$f''(x) = (15x^4 - 15x^2)' = 60x^3 - 30x$$
2. Find the critical points by solving \( f'(x) = 0 \).

\[
15x^4 - 15x^2 = 0
\]

means

\[
15x^2(1 - x^2) = 0
\]

whose roots are \( x = -1, x = 0, x = 1 \).

3. For each of the critical points, plug-them-in \( f''(x) \). If it is positive then it is a local minimum. If it is negative, then it is a local maximum. If it is zero, then you don’t know.

\[
f''(-1) = 60 \cdot (-1)^3 - 30 \cdot (-1) = -30
\]

is negative, so \( x = -1 \) is a local maximum.

\[
f''(0) = 60 \cdot (0)^3 - 30 \cdot (0) = 0
\]

is zero, so for \( x = 0 \) we don’t know

\[
f''(1) = 60 \cdot (1)^3 - 30 \cdot (1) = 30
\]

is positive, so \( x = 1 \) is a local minimum.

Comment: By the First Derivative Test \( f'(x) \) does not change sign when it passes \( x = 0 \), so it is neither max. nor min., but a horizontal inflection point.