

Dr. Z's Math151 Handout #4.3 [The Mean Value Theorem and Monotonicity]

By Doron Zeilberger

Problem Type 4.3.1 : Verify that the function $f(x)$ satisfies the hypothesis of the Mean Value Theorem on the interval $[a, b]$. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Example Problem 4.3.1: Verify that the function $f(x) = x^3 + x - 1$ satisfies the hypothesis of the Mean Value Theorem on the interval $[0, 2]$. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Steps

1. Verify that the function $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) . For polynomials, exponential, and the ‘nice’ trig functions ($\sin x$ and $\cos x$) this is always true. You only have to watch out for rational functions, where the bottom might vanish in the interval, and any expression containing division and/or radicals. For example, for the functions $f(x) = (x - 1)/(x + 2)$ in the interval $[0, 2]$ everything is fine, since the ‘trouble-maker’ $x = -2$ does not lie in the interval $[0, 2]$. But if the interval would have been, say, $[-3, 0]$, then the function is *not* continuous in that interval and the MVT is **not applicable**.

Example

1. $f(x) = x^3 + x - 1$ is a polynomial so it is continuous and differentiable everywhere, and there is nothing to worry about, and MVT can be used.

2. Find $f'(x)$ and set-up the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad .$$

where c is still a symbol, but a and b are the endpoints of the designated interval.

2.

$$f'(x) = (x^3 + x - 1)' = 3x^2 + 1 \quad .$$

The interval is $[a, b] = [0, 2]$ so the equation is

$$3c^2 + 1 = \frac{f(2) - f(0)}{2 - 0} =$$

$$\frac{(2^3 + 2 - 1) - (0^3 + 0 - 1)}{2 - 0} = \frac{10}{2} = 5$$

3. Solve for c the equation from step 2, and retain the number(s) that lie in the interval (a, b) .

3.

$$3c^2 + 1 = 5 \quad ,$$

means that $c^2 = 4/3$, hence

$$c = \pm 2/\sqrt{3} = \pm 2\sqrt{3}/3 \quad .$$

The equation has two roots $c = -2\sqrt{3}/3$, and $c = 2\sqrt{3}/3$, but the former does not belong to $[0, 2]$, while the latter does.

Ans.: $c = 2\sqrt{3}/3$.

Problem Type 4.3.2 : Show that the equation $f(x) = \text{Expression}$ has exactly one root.

Example Problem 4.3.2: Show that the equation $f(x) = 1 + 5x + 4x^3 + 11x^5 + 9x^7$ has exactly one root.

Steps

Example

1. First check that

$$\lim_{x \rightarrow -\infty} f(x) = -\infty \quad ,$$

$$\lim_{x \rightarrow \infty} f(x) = \infty \quad ,$$

or the other way around (i.e. it goes to ∞ when $x \rightarrow -\infty$ and to $-\infty$ when $x \rightarrow \infty$). It follows by the intermediate value theorem that it must cross the x -axis at least once, so there must be *at least* one root to the equation $f(x) = 0$. The present scenario always happens when the degree of the polynomial is an odd integer. If the degree is an even integer, then the present method does not apply.

2. Find $f'(x)$ and try to prove that it is always positive (in the former case) or always negative (in the latter case). This happens, in particular if **all** the terms of $f'(x)$ are of even degree and the coefficients are **all positive**.

3. Conclude by Rolle's Theorem that there can't be more than one real root. Since we know from step 1 that there is at least one real root. We can conclude that there is exactly one root.

1. $f(x)$ is a polynomial of odd degree, hence the conditions apply and $f(x) = 0$ has at least one real root.

2.

$$\begin{aligned} f'(x) &= (1 + 5x + 4x^3 + 11x^5 + 9x^7)' = \\ &5 + 12x^2 + 55x^4 + 63x^6 \quad . \end{aligned}$$

This is obviously always positive.

3. If there would have been two real roots, let's call them $x = a$ and $x = b$ that solve $f(x) = 0$, then it would follow by Rolle's Theorem that there must be a c in (a, b) such that $f'(c) = 0$. But in step 2 we showed that $f'(x)$ is **always** positive, hence this is a contradiction. Since we know from step 1 that there is **at least one** real root, and we have just shown that there is **at most one** real root, we conclude that there is **exactly one** real root.

Problem Type 4.3.3 : If $f(a) = A$ and $f'(x) \geq C$ for $a \leq x \leq b$, how small can $f(b)$ possibly be.

Example Problem 4.3.3: If $f(2) = 3$ and $f'(x) \geq 4$ for $2 \leq x \leq 5$, how small can $f(5)$ possibly be.

Steps

1. By MVT we have that there is a c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} =$$

$$\frac{f(b) - A}{b - a} .$$

But we are told that $f'(x) \geq C$ for *all* x , so in particular for that c , whatever it is. It follows that we have the *inequality*:

$$\frac{f(b) - A}{b - a} \geq C.$$

2. Solve the inequality of step 1 for $f(b)$.

Example

1. By MVT we have that there is a c such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} =$$

$$\frac{f(5) - 3}{3} .$$

But we are told that $f'(x) \geq 4$ for x , so in particular for that c , whatever it is. It follows that we have the *inequality*:

$$\frac{f(5) - 3}{3} \geq 4.$$

2.

$$\frac{f(5) - 3}{3} \geq 4$$

means

$$f(5) - 3 \geq 12 ,$$

which implies

$$f(5) \geq 15 .$$

Ans.: the smallest that $f(5)$ can possibly be is **15**.

Problem from a Previous Final (Spring 2008 #11 (12 points))

Suppose that $f(x)$ is differentiable everywhere and we know that $f(-2) = -1$ and $f'(x) \geq 3$ for all x .

a) (6 points) What is the least possible value for $f(2)$?

b) (6 points) Show that $f(x)$ has a root in $[-2, 2]$.

Solution

(a) By the Mean Value Theorem, there must be a c in the interval $(-2, 2)$ such that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{f(2) - f(-2)}{4} \quad .$$

Since $f(-2) = -1$, this means that

$$f'(c) = \frac{f(2) - (-1)}{4} \quad .$$

We are told that $f'(x) \geq 3$ for **all** x , in particular for our lucky c so we have

$$\frac{f(2) + 1}{4} \geq 3$$

Multiplying both sides by 4 and moving the 1 to the left, gives

$$f(2) \geq 11 \quad .$$

Ans. to (a): The least possible value for $f(2)$ is 11.

b) $f(-2) = -1$ is clearly negative (less than 0). $f(2) \geq 11$ is clearly positive (greater than 0), since 0 is **intermediate**, and the function $f(x)$ is **continuous** (it is even differentiable, so it must be continuous) it follows by the **Intermediate Value Theorem** that somewhere in the interval $(-2, 2)$ the function has the value 0, in other words it must have a **root**.