Problem Type 4.1.1: Use the Linear Approximation to estimate $\Delta f = f(a + h) - f(a)$ for the given function $f(x)$, for the given $a$ and $h$.

Example Problem 4.1.1: Use the Linear Approximation to estimate $\Delta f = f(4 + 0.01) - f(4)$ for $f(x) = 1/x$.

Steps

1. Compute $f'(x)$, and decide who is $a$ and who is $h$.

   1. $f'(x) = -x^{-2} = \frac{-1}{x^2}$, $a = 4$, and $h = 0.01$.

2. Set up the formula $\Delta f \approx f'(a)h$ , and do the plugging-in

   2. $\Delta f \approx f'(4)(0.01) = \frac{-1}{4^2}(0.01) = \frac{-1}{1600}$.

Ans.: $\Delta f \approx \frac{-1}{1600}$

Problem Type 4.1.2: Estimate the quantity using the Linear Approximation.

Example Problem 4.1.2: Estimate the quantity

$$\frac{1}{\sqrt{97}} - \frac{1}{10}.$$

Steps

1. By “pattern-recognition” decide (i) what is the function (ii) what is the “nice” point $a$ (iii) what is the deviation $h$.

   Also find $f'(x)$.

   1. Here $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$. The “nice” point near 97 is $a = 100$ and $h = -3$.

   $$f'(x) = (-1/2)x^{-3/2} = \frac{-1}{2\sqrt{x^3}}.$$
2. Set up the formula

\[ \Delta f \approx f'(a)h , \]

and implement it for the specific problem.

\[ \Delta f \approx f'(100)(-3) = \frac{-1}{2(\sqrt{100})^3}(-3) = \frac{3}{2000} . \]

**Problem Type 4.1.3:** Find the linearization at \( x = a, y = \text{SomeFunction}(x), a = \text{SomeNumber} \).

**Example Problem 4.1.3:** Find the linearization at \( x = a, y = (5 + x^2)^{-1/2}, a = 2 \).

**Steps**

1. Find \( f'(x) \).

\[ f'(x) = \frac{(-1/2)(5+x^2)^{-3/2}((2 \cdot 2) x)}{(\sqrt{5 + 2^2})^3} = \frac{-x}{(\sqrt{5 + 2^2})^3} \]

2. Plug everything into the formula

\[ L(x) = f'(a)(x-a) + f(a) , \]

\[ L(x) = f'(a)(x-2)+f(2) = \frac{-2}{(\sqrt{5 + 2^2})^3}(x-2) - (5+2^2)^{-1/2} \]

\[ = \frac{-2}{27}(x-2) + (9)^{-1/2} = \frac{-2}{27}(x-2) + \frac{1}{3} . \]

**Ans.:** \( L(x) = \frac{-2}{27}(x-2) + \frac{1}{3} \).
Problem from a Previous Final (Spring 2008, #4 (9 points)).

Let \( f(x) = \sqrt{1 - x} \)

(a) (6 points) Using the linear approximation of \( f(x) \) at \( a = -3 \) compute an approximation to \( f(-4) \).

(b) (3 points) Use \( f'' \) (concavity) to determine whether your approximation is larger or smaller than the true value of \( f(-4) \).

Solution

(a) \( f(x) = \sqrt{1 - x} = (1 - x)^{1/2} \). Since \( a = -3 \), \( f(-3) = 4^{1/2} = 2 \). We also have

\[
\frac{d}{dx} f(x) = \frac{1}{2}(1 - x)^{-1/2} = \frac{-1}{2\sqrt{1 - x}}.
\]

Plugging-in \( x = -3 \) gives

\[
f'(-3) = \frac{-1}{2\sqrt{1 - (-3)}} = \frac{-1}{2\sqrt{4}} = \frac{-1}{4}.
\]

The linear approximation is

\[
L(x) = f'(-3)(x - (-3)) + f(-3) = \frac{-1}{4} (x + 3) + 2.
\]

And when \( x = -4 \), we get

\[
L(-4) = \frac{-1}{4} (-4 + 3) + 2 = 9/4.
\]

Ans. to (a): The approximation to \( f(-4) \) using the Linear approximation is \( 9/4 = 2.25 \).

(b) \( f''(x) = (-1/4)(1 - x)^{-3/2} \). So \( f''(-4) = (-1/4) \cdot 5^{3/2} \) is negative, this means that the approximation is larger than the true value of \( f(-4) \).