Dr. Z's Math151 Handout #3.9 [Derivatives of Inverse Functions]

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Problem Type 3.9.1 : Use the formula

$$g'(x) = \frac{1}{f'(g(x))} \quad ,$$

to calculate $g'(x)$, where $g(x)$ is the inverse of the given function $f(x)$).

Example Problem 3.9.2: Use the formula

$$g'(x) = \frac{1}{f'(g(x))} \quad ,$$

to calculate $g'(x)$, where $g(x)$ is the inverse of $f(x) = \sqrt{3-x}$.

Steps

1. Use algebra to find the inverse function, g(x), by solving $y = f(x)$ for x, getting an expression in y, and replacing y by x.

Example

1. We have to solve

$$y = \sqrt{3-x} \quad .$$

Squaring both sides gives:

$$y^2 = 3 - x \quad ,$$

and solving for x, in terms of y, gives the expression $x = 3 - y^2$. Replacing y by x gives $g(x) = 3 - x^2$.

2. Find $f'(x)$, and plug it into

$$g'(x) = \frac{1}{f'(g(x))} \quad ,$$

2.

$$f'(x) = ((3-x)^{1/2})' = (1/2)(3-x)^{-1/2}(-1) = \frac{-1}{2\sqrt{3-x}} \quad .$$

Now using

$$g'(x) = \frac{1}{f'(g(x))} \quad ,$$

we get

$$g'(x) = \frac{-1}{\frac{1}{2\sqrt{3-(3-x^2)}}}$$

$$= \frac{-1}{\frac{1}{2\sqrt{x^2}}} = \frac{-1}{\frac{1}{2x}} = -2x$$

Ans.: $g'(x) = -2x$.

Comment: That was a very stupid way to compute $g'(x)$! $g(x) = 3 - x^2$ is much simpler than $f(x)$ and it is much more efficient, in this case, to do it directly: $g'(x) = -2x$.

Problem Type 3.9.2 : Find the derivative of an expression involving one of the inverse trig functions $\sin^{-1}$, $\cos^{-1}$,$\tan^{-1}$, etc.

Example Problem 3.9.2: Find the derivative of $y = \cos^{-1}(3x + 1)$.

Steps

1. Use all applicable differentiation rules, remembering the derivatives of the new functions:

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \quad ,$$

$$\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}} \quad ,$$

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2} \quad .$$

Example

1. By the chain rule

$$\frac{d}{dx}(\cos^{-1}(3x+1)) = -\frac{1}{\sqrt{1-(3x+1)^2}} \cdot \frac{d}{dx}(3x+1)$$

2. Simplify

2.

$$-\frac{3}{\sqrt{1-(9x^2+6x+1)}} =$$

$$-\frac{3}{\sqrt{-9x^2-6x}} =$$

$$-\frac{3}{\sqrt{-3x(3x+2)}} \quad .$$

Problem from a Previous Final Exam (Spring 2008 #12b (5 points)).

Use rules of differentiation to calculate $f'(x)$ (Do not simplify your answer).

$$f(x) = (\cos^{-1} x)^{\frac{3}{2}} \cdot \ln\left(\frac{x}{x+1}\right)$$

Solution. Remember the **Golden Rule**: *Simplify before you differentiate!*. So we first use the ln rule $\ln(A/B) = \ln A - \ln B$ to write

$$f(x) = (\cos^{-1} x)^{\frac{3}{2}} \cdot (\ln x - \ln(x+1)) \quad .$$

Next we use the **prodcut rule**.

$$f'(x) = [(\cos^{-1} x)^{\frac{3}{2}}]' \cdot [\ln x - \ln(x+1)] + [(\cos^{-1} x)^{\frac{3}{2}}] \cdot [\ln x - \ln(x+1)]' \quad .$$

Next we use the **chain rule** (and $(\ln x)' = 1/x$)):

$$= [\frac{3}{2}(\cos^{-1} x)^{\frac{1}{2}}] \cdot (\cos^{-1} x)' \cdot [\ln x - \ln(x+1)] + [(\cos^{-1} x)^{\frac{3}{2}}] \cdot [\frac{1}{x} - \frac{1}{x+1}] \quad .$$

Finally we use

$$(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}} \quad ,$$

and get

$$= [\frac{3}{2}(\cos^{-1} x)^{\frac{1}{2}}] \cdot \frac{-1}{\sqrt{1-x^2}} \cdot [\ln x - \ln(x+1)] + [(\cos^{-1} x)^{\frac{3}{2}}] \cdot [\frac{1}{x} - \frac{1}{x+1}] \quad .$$

This is the answer!. Do not dare simplify!